

SPECTRAL AND SCATTERING THEORY OF SELF-ADJOINT HANKEL OPERATORS WITH PIECEWISE CONTINUOUS SYMBOLS

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ABSTRACT. We develop the spectral and scattering theory for self-adjoint Hankel operators H with piecewise continuous symbols. In this case every jump of the symbol gives rise to a band of the absolutely continuous spectrum of H . We construct wave operators relating simple “model” (that is, explicitly diagonalizable) Hankel operators for each jump and the given Hankel operator H . We show that the set of all these wave operators is asymptotically complete. This determines the absolutely continuous part of H . We also prove that the singular continuous spectrum of H is empty and that its eigenvalues may accumulate only to “thresholds” in the absolutely continuous spectrum. All these results are reformulated in terms of Hankel operators realized as matrix or integral operators.

1. INTRODUCTION

1.1. Hankel operators (see the books [10, 11, 13]) admit various unitary equivalent descriptions. One of the common ones is the definition of Hankel operators H in the Hardy space $\mathbb{H}_+^2(\mathbb{T}) \subset L^2(\mathbb{T})$ of functions analytic inside the unit circle \mathbb{T} . Let $\omega \in L^\infty(\mathbb{T})$. Then for $f \in \mathbb{H}_+^2(\mathbb{T})$, the function $(Hf)(\mu)$, $\mu \in \mathbb{T}$, is defined as the orthogonal projection in $L^2(\mathbb{T})$ of the function $\omega(\mu)f(\bar{\mu})$ onto the subspace $\mathbb{H}_+^2(\mathbb{T})$. Of course, Hankel operators $H = H(\omega)$ with symbols $\omega \in L^\infty(\mathbb{T})$ are bounded.

It is easy to see that H is compact if $\omega \in C(\mathbb{T})$. On the contrary, the jumps

$$\varkappa(a) = \lim_{\varepsilon \rightarrow +0} \omega(ae^{i\varepsilon}) - \lim_{\varepsilon \rightarrow +0} \omega(ae^{-i\varepsilon}), \quad a \in \mathbb{T}, \quad (1.1)$$

(one supposes here that the limits exist but are not equal) of the symbol yield bands of the essential spectrum $\text{spec}_{\text{ess}}(H)$. To be more precise, it was shown by S. R. Power

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in [12, 13] that

$$\text{spec}_{\text{ess}}(H) = [0, (2i)^{-1}\kappa(1)] \cup [0, (2i)^{-1}\kappa(-1)] \cup \bigcup_{\text{Im } a_j > 0} [-(2i)^{-1}(\kappa(a_j)\kappa(\bar{a}_j))^{1/2}, (2i)^{-1}(\kappa(a_j)\kappa(\bar{a}_j))^{1/2}]$$

where $[\alpha, \beta]$ is the interval between the points $\alpha, \beta \in \mathbb{C}$ (we do not distinguish $[\alpha, \beta]$ and $[\beta, \alpha]$). Note that the contribution of a jump at a complex point a is nontrivial only if the symbol $\omega(\mu)$ has jumps at both points a and \bar{a} .

Much more complete information (see subs. 3.4, for more details) can be obtained about the modulus $|H| = \sqrt{H^*H}$ of H . It is shown in [15] that the absolutely continuous (a.c.) spectrum¹ of $|H|$

$$\text{spec}_{\text{ac}}(|H|) = \bigcup_{a_j \in \mathbb{T}} [0, 2^{-1}|\kappa(a_j)|]. \quad (1.2)$$

It is assumed in [15] that $\omega(\mu)$ has finitely many jumps $a_j \in \mathbb{T}$ so that the union in (1.2) also has finite number of terms. Furthermore, the singular continuous spectrum of $|H|$ is empty and its eigenvalues different from 0 and the points $2^{-1}|\kappa(a_j)|$ have finite multiplicities and may accumulate only to these points.

1.2. Our goal here is to develop the spectral and scattering theory for *self-adjoint* Hankel operators with piecewise continuous symbols. In the self-adjoint case we have $\omega(\bar{\mu}) = \overline{\omega(\mu)}$. Therefore if $\omega(\mu)$ has a jump $\kappa(a)$ at some point $a \in \mathbb{T}$, then it also has the jump $\kappa(\bar{a}) = -\overline{\kappa(a)}$ at the point \bar{a} ; in particular, $\kappa(\pm 1)$ are necessarily imaginary numbers. We suppose that $\omega(\mu)$ has a finite number of jumps and that $\omega(\mu)$ is a continuous function away from its jumps. We assume that at every jump a , the one-sided limits $\omega(a_{\pm})$ of $\omega(\mu)$ as $\mu \rightarrow a_{\pm} = ae^{\pm i0}$ exist and satisfy the logarithmic Hölder continuity condition

$$\omega(\mu) - \omega(a_{\pm}) = O(|\ln |\mu - a||^{-\beta_0}), \quad \beta_0 > 0. \quad (1.3)$$

This condition is slightly stronger than the simple one-sided continuity of $\omega(\mu)$ but is of course weaker than the Hölder continuity. One of our main results (see Theorem 6.2) can be formulated as follows.

¹*If $\beta_0 > 1$, then*

$$\text{spec}_{\text{ac}}(H) = [0, (2i)^{-1}\kappa(1)] \cup [0, (2i)^{-1}\kappa(-1)] \cup \bigcup_{\text{Im } a_j > 0} [-2^{-1}|\kappa(a_j)|, 2^{-1}|\kappa(a_j)|]. \quad (1.4)$$

²*If $\beta_0 > 2$, then the singular continuous spectrum of H is empty and its eigenvalues distinct from 0 and from the points $(2i)^{-1}\kappa(1)$, $(2i)^{-1}\kappa(-1)$ and $\pm 2^{-1}|\kappa(a_j)|$ where $\text{Im } a_j > 0$ have finite multiplicities and may accumulate only to these points.*

¹In the right-hand side of (1.2) as well as in all relations of this type, we use the convention that every term denotes a spectral band of multiplicity one in the a.c. spectrum.

Relation (1.4) shows that every real jump or a pair of complex conjugate jumps of the symbol $\omega(\mu)$ yields its own band of the multiplicity one a.c. spectrum. Of course if $\varkappa(1) = 0$ or $\varkappa(-1) = 0$, then the corresponding term in (1.4) disappears.

Under different assumptions on $\omega(\mu)$, relation (1.4) was obtained earlier by J. S. Howland in [4] who used the trace class method in scattering theory. We rely on a multi-channel scheme in the so-called smooth approach exposed in our preceding publication [14]. The results of [14] can be considered as a simplified version of the famous Faddeev's solution [2] of the three particle quantum problem. In our case different channels of scattering are described in terms of "model" operators for all points of discontinuity of ω . Each model operator H_+ , H_- and H_j corresponds to jumps of ω at the points $+1$, -1 and (a_j, \bar{a}_j) , $\text{Im } a_j > 0$, and each of them yields one of the bands of the a.c. spectrum in the right-hand side of (1.4).

Our conditions on the symbol $\omega(\mu)$ are much weaker than those of J. S. Howland. This is important for reformulations of our results for Hankel operators realized as matrix and integral operators.

1.3. Our approach shows that the resolvent of the operator H sandwiched between appropriate weight functions has boundary values on the real axis except for the thresholds 0 , $(2i)^{-1}\varkappa(1)$, $(2i)^{-1}\varkappa(-1)$ and $\pm 2^{-1}|\varkappa(a_j)|$ for all complex points of jumps a_j . Results of this type are known as the limiting absorption principle.

Another important ingredient of our approach is the construction of the wave operators for model operators H_+ , H_- , H_j and the operator H . They play the same role as the wave operators for different channels of scattering in the three particle problem. Thus in the case of several jumps of the symbol, the scattering problem for Hankel operators becomes multichannel. Similarly to the three particle problem, the ranges of different wave operators are orthogonal to each other, and their orthogonal sum coincides with the a.c. subspace of the operator H . The last result is known as the asymptotic completeness of wave operators. It directly implies relation (1.4) but also contains information about the asymptotics of $\exp(-iHt)f$ as $t \rightarrow \pm\infty$.

The basis for the construction of model operators is an explicit diagonalization of some simple Hankel operator. We recall that Hankel operators can be realized as integral operators in the space $L^2(\mathbb{R}_+)$ with kernels $\mathbf{h}(t+s)$ which depend only on the sum of variables $t, s \in \mathbb{R}_+$. We rely on the Hankel operator \mathcal{M} with $\mathbf{h}(t) = \pi^{-1}(t+2)^{-1}$ considered by F. G. Mehler in [8]. Due to a slow decay of this function as $t \rightarrow \infty$ the operator \mathcal{M} is not compact. Actually, it has the simple purely a.c. spectrum coinciding with the interval $[0, 1]$. In terms of the symbol $\psi(\nu)$ (the Fourier transform of $(2\pi)^{1/2}\mathbf{h}(t)$), this fact is a consequence (cf. relation (1.4)) of the jump $2i$ of $\psi(\nu)$ at the point $\nu = 0$.

Model operators H_+ and H_- are directly constructed in terms of the operator \mathcal{M} . The construction of model operators H_j for pairs of complex jumps is much more complicated. The operators H_j can also be realized as Hankel operators but their

symbols are 2×2 matrix valued functions so that they act in the direct sum $\mathbb{H}_+^2(\mathbb{T}) \oplus \mathbb{H}_-^2(\mathbb{T})$.

The construction of scattering theory for the pairs (H_+, H) , (H_-, H) and (H_j, H) relies on the following arguments. We show that with an appropriate choice of model operators the symbol of the operator

$$\tilde{H} = H - H_+ - H_- - \sum_{\operatorname{Im} a_j > 0} H_j$$

has no jumps and so it is, in some sense, negligible. To be more precise, we establish a factorization $\tilde{H} = Q^* \tilde{K} Q$ where \tilde{K} is compact and Q is smooth with respect to all operators H_+ , H_- and H_j . Roughly speaking, this means that $Q\vartheta \in L^2(\mathbb{T})$ for all eigenfunctions ϑ (of the continuous spectra) of H_+ , H_- and H_j . Similar factorizations are true for all products H_+H_- , $H_\pm H_j$ and $H_j H_k$ where $j \neq k$. Here we use the fact that the singularities of the symbols of the operators H_+ , H_- and H_j are disjoint. These analytic results allow us to verify the assumptions of [14] which, in particular, yields spectral results on the operator H stated above.

Note that a similar scheme has been used by S. R. Power in his study of the essential spectrum of H in [12, 13] where it was however sufficient to verify the compactness of the operators \tilde{H} , H_+H_- , $H_\pm H_j$ and $H_j H_k$ where $j \neq k$. For the study of the a.c. spectrum of H , J. S. Howland required in [4] that these operators belong to the trace class.

An important issue in our approach is the choice of the class of smooth operators Q . As Q , we choose the operator of multiplication by a function $q(\mu)$ vanishing in singular points of the symbol $\omega(\mu)$. This choice is well adapted to the separation of singularities of $\omega(\mu)$. Another possibility (see [19]) is to choose for Q the operator of multiplication by a function of t tending to zero as $t \rightarrow 0$ and $t \rightarrow \infty$ in the realization of Hankel operators as integral operators in $L^2(\mathbb{R}_+)$.

1.4. Let us now state our results for Hankel operators \hat{H} in the space $\ell_+^2 = \ell^2(\mathbb{Z}_+)$ where \hat{H} is defined by the formula

$$(\hat{H}u)_n = \sum_{m=0}^{\infty} h_{n+m} u_m, \quad u = (u_0, u_1, \dots) \in \ell_+^2, \quad h_n = \bar{h}_n, \quad \mathbb{Z}_+. \quad (1.5)$$

Recall that \hat{H} is compact if $h_n = o(n^{-1})$ as $n \rightarrow \infty$. On the other hand, if $h_n = \pi^{-1}(n+1)^{-1}$, then \hat{H} (the Hilbert matrix) has the simple a.c. spectrum coinciding with $[0, 1]$. We assume that

$$h_n = (\pi n)^{-1} (\kappa_+ + (-1)^n \kappa_- + 2 \sum_{j=1}^{N_0} \kappa_j \sin(n\theta_j - \varphi_j)) + O(n^{-1}(\ln n)^{-\alpha_0}) \quad (1.6)$$

as $n \rightarrow \infty$. Here θ_j are distinct numbers in $(0, \pi)$; the phases $\varphi_j \in [0, \pi)$ and the amplitudes $\kappa_+, \kappa_-, \kappa_j \in \mathbb{R}$ are arbitrary. The main result in this case is (see Theorem 7.5):

¹ If $\alpha_0 > 2$, then

$$\text{spec}_{\text{ac}}(\widehat{H}) = [0, \kappa_+] \cup [0, \kappa_-] \cup \bigcup_{j=1}^{N_0} [-\kappa_j, \kappa_j]. \quad (1.7)$$

² If $\alpha_0 > 3$, then the singular continuous spectrum of \widehat{H} is empty and its eigenvalues different from 0 and the points κ_+ , κ_- and $\pm\kappa_j$ have finite multiplicities and may accumulate only to these points.

Observe that the right-hand side of (1.6) contains oscillations with different frequencies. Formula (1.7) shows that every term in asymptotics (1.6) yields its own channel of scattering and that there is no “interference” between different terms.

Note that the condition $O(n^{-1}(\ln n)^{-\alpha_0})$ on matrix elements does not guarantee (for any α_0) that the corresponding Hankel operator belongs to the trace class. So J. S. Howland’s results [4] do not cover the case of matrix elements with asymptotics (1.6).

1.5. The paper is organized as follows. In Section 2 we recall the results of our preceding publication [14] on multichannel scheme in scattering theory. Auxiliary information about Hankel operators is collected in Section 3. This information is used to state our results in different representations of Hankel operators.

Main results are stated and proven in Section 6. The proofs consist of 2 ingredients. The first is the construction of “model” operators for all points of discontinuity of ω . This is carried out in Section 4 for real points and in Section 5 for pairs of complex conjugate points. Here the smoothness of some class of operators with respect to model operators is also verified. The second ingredient consists of compactness results about Hankel operators sandwiched by singular weights. We borrow these results from our previous publication [15]. All our results can be extended to Hankel operators acting in the Hardy space $\mathbb{H}_+^2(\mathbb{R})$ and to operators with operator-valued symbols.

Finally, in Section 7 we state our results for Hankel operators realized as infinite matrices in the space ℓ_+^2 and as integral operators in the space $L^2(\mathbb{R}_+)$.

2. MULTICHANNEL SCHEME

In the first two subsections, we collect some background facts from scattering theory; see, e.g., the book [17], for a detailed presentation. In the last subsection we recall the results of our paper [14] which will be used here.

2.1. Let H be a self-adjoint operator in a Hilbert space \mathcal{H} , and let $E(\cdot) = E(\cdot; H)$ be its spectral family. We denote by $\mathcal{H}^{(\text{p})}(H)$ the subspace of \mathcal{H} spanned by all eigenvectors of the operator H and by $\mathcal{H}^{(\text{ac})}(H)$ its a.c. subspace; $P^{(\text{ac})}(H)$ is the orthogonal projector onto $\mathcal{H}^{(\text{ac})}(H)$; $H^{(\text{ac})}$ is the restriction of H onto $\mathcal{H}^{(\text{ac})}(H)$.

Suppose that the spectrum of the operator H is a.c. and has a constant (possibly infinite) multiplicity n on a bounded open interval $\Delta \subset \mathbb{R}$. We consider a unitary

mapping

$$F : E(\Delta)\mathcal{H} \rightarrow L^2(\Delta; \mathcal{N}) = L^2(\Delta) \otimes \mathcal{N}, \quad \dim \mathcal{N} = n, \quad (2.1)$$

of the subspace $E(\Delta)\mathcal{H}$ onto the space of vector-valued functions of $\lambda \in \Delta$ with values in \mathcal{N} . Assume that this mapping transforms H into the operator A_Δ of multiplication by λ in the space $L^2(\Delta; \mathcal{N})$, that is,

$$(FHf)(\lambda) = \lambda(Ff)(\lambda), \quad f \in E(\Delta)\mathcal{H}, \quad \lambda \in \Delta.$$

Along with $L^2(\Delta; \mathcal{N})$, we consider the space $C^\gamma(\Delta; \mathcal{N})$, $\gamma \in (0, 1]$, of Hölder continuous vector-valued functions. We set $Ff = 0$ for $f \in E(\mathbb{R} \setminus \Delta)\mathcal{H}$.

Definition 2.1. Let Q be a bounded operator in the space \mathcal{H} . The operator Q is called strongly H -smooth on an interval Δ with an exponent $\gamma \in (0, 1]$ if, for some diagonalization F of the operator $E(\Delta)H$, the condition

$$|(FQ^*f)(\lambda)| \leq C\|f\|, \quad |(FQ^*f)(\lambda') - (FQ^*f)(\lambda)| \leq C|\lambda' - \lambda|^\gamma\|f\| \quad (2.2)$$

is satisfied for all $f \in \mathcal{H}$. Here the constant C does not depend on λ and λ' in compact subintervals of Δ .

Definition 2.1 depends on the choice of mapping (2.1), but in applications the operator F emerges naturally. For a strongly H -smooth operator Q , the operator $Z(\lambda; Q) : \mathcal{H} \rightarrow \mathcal{N}$, defined by the relation

$$Z(\lambda; Q)f = (FQ^*f)(\lambda),$$

is bounded and depends Hölder continuously on $\lambda \in \Delta$.

Assume that an operator Q is strongly H -smooth. If an operator B is bounded, then the product BQ is also strongly H -smooth. Let U be a unitary operator in \mathcal{H} and $\tilde{H} = U^*HU$. Then the operator $\tilde{Q} = QU$ is strongly \tilde{H} -smooth for the diagonalization $\tilde{F} = FU$ of \tilde{H} .

It is convenient to give also a “global” definition of H -smoothness adapted to our purposes.

Definition 2.2. Suppose that, apart from the point spectrum $\text{spec}_p(H)$, the spectrum of a self-adjoint operator H is a.c., has a constant multiplicity and coincides with the closure of a finite union Δ of disjoint open intervals $\Delta^{(l)} = (\alpha_l, \beta_l)$, $l = 1, \dots, L$. Assume also that $\text{spec}_p(H) \cap \Delta = \emptyset$. A bounded operator Q is called strongly H -smooth if it is strongly H -smooth on all intervals $\Delta^{(l)}$.

Under the hypothesis of the above definition, the set \mathcal{T} of thresholds of the operator H is defined as the collection of all end points $\alpha_1, \beta_1, \dots, \alpha_L, \beta_L$.

2.2. Suppose that for self-adjoint operators H_0 and H , the strong limits

$$\text{s-lim}_{t \rightarrow \pm\infty} e^{iHt} e^{-iH_0 t} P^{(\text{ac})}(H_0) =: W_\pm(H, H_0)$$

exist. The operators $W_{\pm}(H, H_0)$ are known as wave operators. They are automatically isometric on the subspace $\mathcal{H}^{(\text{ac})}(H_0)$, enjoy the intertwining property

$$HW_{\pm}(H, H_0) = W_{\pm}(H, H_0)H_0,$$

and their ranges $R(W_{\pm}(H, H_0)) \subset \mathcal{H}^{(\text{ac})}(H)$.

The wave operator $W_{\pm}(H, H_0)$ is called complete if $R(W_{\pm}(H, H_0)) = \mathcal{H}^{(\text{ac})}(H)$. The completeness of $W_{\pm}(H, H_0)$ is equivalent to the existence of $W_{\pm}(H_0, H)$; in this case

$$W_{\pm}(H_0, H) = W_{\pm}^*(H, H_0).$$

Note also the multiplication theorem: if the wave operators $W_{\pm}(H, H_1)$ and $W_{\pm}(H_1, H_0)$ exist, then the wave operator $W_{\pm}(H, H_0)$ also exists and

$$W_{\pm}(H, H_0) = W_{\pm}(H, H_1)W_{\pm}(H_1, H_0). \quad (2.3)$$

2.3. Let H_1, \dots, H_N and \tilde{H} be bounded self-adjoint operators on a Hilbert space \mathcal{H} . Our goal is to study the spectral properties of the operator

$$H = H_1 + \dots + H_N + \tilde{H} \quad (2.4)$$

under certain smoothness assumptions on all products $H_n H_m$, $n \neq m$, and on the operator \tilde{H} . We suppose that all operators H_n , $n = 1, \dots, N$, satisfy the conditions of Definition 2.2 on a set Δ_n and that $0 \notin \Delta_n$. Note that in interesting cases 0 belongs to the closure of Δ_n at least for one n . Let Q be a bounded operator on \mathcal{H} such that its kernel is trivial and its range $R(Q)$ is dense in \mathcal{H} . We need the following

Assumption 2.3. *a. For all $n = 1, \dots, N$, the operator Q is strongly H_n -smooth (see Definitions 2.1 and 2.2) with an exponent $\gamma \in (0, 1]$.*

b. The operator \tilde{H} can be represented as $\tilde{H} = Q^ \tilde{K} Q$ with a compact operator \tilde{K} .*

c. For all $n, m \geq 1$, $n \neq m$, the operators $H_n H_m$ can be represented as

$$H_n H_m = Q^* K_{n,m} Q$$

where the operators $K_{n,m}$ are compact.

d. For all $n = 1, \dots, N$, the operators $Q H_n Q^{-1}$ defined on the set $R(Q)$ extend to bounded operators.

The spectral structure of the operator (2.4) is described in the following assertion. We denote by \mathcal{T}_n be the set of the thresholds of the operator H_n and put $\mathcal{T} = \mathcal{T}_1 \cup \dots \cup \mathcal{T}_N$. Recall that by definition, A_{Δ} is the operator of multiplication by the independent variable in $L^2(\Delta)$.

Theorem 2.4. [14] *Under Assumption 2.3 we have:*

¹⁰ *The operator $H^{(\text{ac})}$ is unitarily equivalent to the direct sum*

$$A_{\Delta_1} \oplus \dots \oplus A_{\Delta_N}.$$

2⁰ Suppose additionally that $\gamma > 1/2$. Then the singular continuous spectrum of H is empty and the eigenvalues of H in the set $\mathbb{R} \setminus \mathcal{T}$ have finite multiplicities and can accumulate only to the set \mathcal{T} .

The following assertion is known as the limiting absorption principle.

Theorem 2.5. [14] *Under Assumption 2.3 with $\gamma > 1/2$, the operator-valued function $Q(H - z)^{-1}Q^*$ is Hölder continuous with any exponent $\gamma' < \gamma$ in z if $\pm \operatorname{Im} z \geq 0$, $\operatorname{Re} z \in \mathbb{R} \setminus \mathcal{T}$ away from eigenvalues of H .*

The following assertion summarizes the scattering theory for the set of the operators H_1, \dots, H_N and the operator H .

Theorem 2.6. [14] *Under Assumption 2.3 we have:*

1⁰ *For all $n = 1, \dots, N$, the wave operators $W_{\pm}(H, H_n)$ exist.*

2⁰ *These operators enjoy the intertwining property*

$$HW_{\pm}(H, H_n) = W_{\pm}(H, H_n)H_n, \quad n = 1, \dots, N.$$

The wave operators are isometric and their ranges are orthogonal to each other, that is,

$$\operatorname{Ran} W_{\pm}(H, H_n) \perp \operatorname{Ran} W_{\pm}(H, H_m), \quad n \neq m.$$

3⁰ *The asymptotic completeness holds:*

$$\operatorname{Ran} W_{\pm}(H, H_1) \oplus \dots \oplus \operatorname{Ran} W_{\pm}(H, H_N) = \mathcal{H}^{(\text{ac})}(H).$$

3. HANKEL OPERATORS

Here we collect standard information on various representations (see the diagrams below) of Hankel operators H . Observe that Hankel operators are always defined by the same formula

$$H = P_+ \Omega J P_+^*, \tag{3.1}$$

but the definitions of the operators P_+ , Ω and J depend on the representation. We will consider four representations and state our results in all of them. It is convenient to keep in mind all representations because some results obvious in one of them are difficult to see in others.

3.1. Let us begin with the representation of Hankel operators in the Hardy space $\mathbb{H}_+^2(\mathbb{T}) \subset L^2(\mathbb{T})$ of functions analytic in the unit disc \mathbb{D} (for the precise definitions of Hardy classes, see, e.g., the book [3]). The norm in the space $L^2(\mathbb{T})$ is defined in a standard way by

$$\|f\|_{L^2(\mathbb{T})}^2 = \int_{\mathbb{T}} |f(\mu)|^2 dm(\mu), \quad dm(\mu) = (2\pi i \mu)^{-1} d\mu.$$

Note that $dm(\mu)$ is the Lebesgue measure on \mathbb{T} normalized so that $m(\mathbb{T}) = 1$. In formula (3.1), $P_+ : L^2(\mathbb{T}) \rightarrow \mathbb{H}_+^2(\mathbb{T})$ is the orthogonal projection, $J = J^* : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ is the involution,

$$(Jf)(\mu) = \bar{\mu}f(\bar{\mu}), \quad \mu \in \mathbb{T}.$$

Obviously, J maps $\mathbb{H}_+^2(\mathbb{T})$ onto $\mathbb{H}_-^2(\mathbb{T})$ where $\mathbb{H}_-^2(\mathbb{T}) = \mathbb{H}_+^2(\mathbb{T})^\perp$ is the space of functions analytic in $\mathbb{C} \setminus \overline{\mathbb{D}}$ and decaying at infinity. The operator $P_- = JP_+J$ is the orthogonal projection of $L^2(\mathbb{T})$ onto $\mathbb{H}_-^2(\mathbb{T})$.

The operator of multiplication $\Omega : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ is defined by the formula

$$(\Omega f)(\mu) = \mu \omega(\mu) f(\mu), \quad \mu \in \mathbb{T}. \quad (3.2)$$

The function $\omega(\mu)$ is always assumed to be bounded. Thus operator (3.1) is determined by the function $\omega(\mu)$, that is, $H = H(\omega)$. The function $\omega(\mu)$ is known as the symbol of the Hankel operator $H(\omega)$. Of course the symbol is not unique because $H(\omega_1) = H(\omega_2)$ if (and only if) $\omega_1 - \omega_2 \in \mathbb{H}_-^\infty(\mathbb{T})$.

We set $\ell^2 = \ell^2(\mathbb{Z})$ and $\ell_+^2 = \ell^2(\mathbb{Z}_+)$ where $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. The unitary mapping $\mathcal{F} : L^2(\mathbb{T}) \rightarrow \ell^2$ corresponds to expanding a function in the Fourier series:

$$\hat{f}_n = (\mathcal{F}f)_n = \int_{\mathbb{T}} f(\mu) \mu^{-n} dm(\mu) \quad (3.3)$$

so that for a sequence $\hat{f} = \{\hat{f}_n\}$, $n \in \mathbb{Z}$,

$$f(\mu) = (\mathcal{F}^* \hat{f})(\mu) = \sum_{n=-\infty}^{\infty} \hat{f}_n \mu^n. \quad (3.4)$$

Then $\hat{P}_+ = \mathcal{F}P_+\mathcal{F}^* : \ell^2 \rightarrow \ell_+^2$ is the orthogonal projection onto the subspace ℓ_+^2 . The operators $\hat{J} = \mathcal{F}J\mathcal{F}^* : \ell^2 \rightarrow \ell^2$ and $\hat{\Omega} = \mathcal{F}\Omega\mathcal{F}^* : \ell^2 \rightarrow \ell^2$ act by the formulas

$$(\hat{J}\hat{f})_n = \hat{f}_{-n-1}$$

and

$$(\hat{\Omega}\hat{f})_n = \sum_{m=-\infty}^{\infty} \hat{\omega}_{n-m-1} \hat{f}_m$$

where $\hat{\omega}_n$ are the Fourier coefficients of the function $\omega(\mu)$. According to (3.1), this leads to the standard definition of the Hankel operator

$$\hat{H} = \mathcal{F}H\mathcal{F}^* = \hat{P}_+ \hat{\Omega} \hat{J} \hat{P}_+^* : \ell_+^2 \rightarrow \ell_+^2$$

by the formula

$$(\hat{H}\hat{f})_n = \sum_{m=0}^{\infty} \hat{\omega}_{n+m} \hat{f}_m.$$

3.2. Recall that the mapping

$$\mu = \frac{\nu - i/2}{\nu + i/2}$$

of \mathbb{R} onto \mathbb{T} can be extended to the conformal mapping from the upper half-plane onto the unit disc. The unitary operator $\mathcal{U} : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{R})$ corresponding to this mapping is defined by the equality

$$(\mathcal{U}f)(\nu) = (2\pi)^{-1/2} (\nu + i/2)^{-1} f\left(\frac{\nu - i/2}{\nu + i/2}\right). \quad (3.5)$$

Since

$$\nu = \frac{i}{2} \frac{1+\mu}{1-\mu},$$

we have

$$(\mathcal{U}^* \mathbf{f})(\mu) = i(2\pi)^{1/2} (1-\mu)^{-1} \mathbf{f}\left(\frac{i}{2} \frac{1+\mu}{1-\mu}\right). \quad (3.6)$$

Observe that $\mathcal{U} : \mathbb{H}_\pm^2(\mathbb{T}) \rightarrow \mathbb{H}_\pm^2(\mathbb{R})$ and that $\mathbf{P}_\pm = \mathcal{U} P_\pm \mathcal{U}^*$ are the orthogonal projections onto the Hardy classes $\mathbb{H}_\pm^2(\mathbb{R})$. Set $\mathbf{J} = -\mathcal{U} J \mathcal{U}^*$, $\mathbf{\Omega} = -\mathcal{U} \Omega \mathcal{U}^*$. Then $(\mathbf{J}\mathbf{f})(\nu) = \mathbf{f}(-\nu)$ and

$$(\mathbf{\Omega}\mathbf{f})(\nu) = \psi(\nu) \mathbf{f}(\nu)$$

where

$$\psi(\nu) = -\frac{\nu-i/2}{\nu+i/2} \omega\left(\frac{\nu-i/2}{\nu+i/2}\right). \quad (3.7)$$

As always,

$$\mathbf{H} = \mathbf{H}(\psi) = \mathcal{U} H(\omega) \mathcal{U}^* = \mathbf{P}_+ \mathbf{\Omega} \mathbf{J} \mathbf{P}_+^*. \quad (3.8)$$

The last, fourth, representation is obtained by applying the Fourier transform Φ :

$$\hat{\mathbf{f}}(t) = (\Phi \mathbf{f})(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \mathbf{f}(\nu) e^{-i\nu t} d\nu.$$

Then $\hat{\mathbf{P}}_\pm = \Phi \mathbf{P}_\pm \Phi^*$ acts as the multiplication by the characteristic function $\mathbb{1}_\pm$ of the half-axis \mathbb{R}_\pm , that is,

$$(\hat{\mathbf{P}}_\pm \hat{\mathbf{f}})(t) = \mathbb{1}_\pm(t) \hat{\mathbf{f}}(t).$$

In this representation,

$$(\hat{\mathbf{J}}\hat{\mathbf{f}})(t) = (\Phi \mathbf{J} \Phi^* \hat{\mathbf{f}})(t) = \hat{\mathbf{f}}(-t)$$

and $\hat{\mathbf{\Omega}} = \Phi^* \mathbf{\Omega} \Phi$ is the convolution:

$$(\hat{\mathbf{\Omega}}\hat{\mathbf{f}})(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{\psi}(t-s) \hat{\mathbf{f}}(s) ds. \quad (3.9)$$

Then the Hankel operator

$$\hat{\mathbf{H}} = \Phi \mathbf{H} \Phi^* = \hat{\mathbf{P}}_+ \hat{\mathbf{\Omega}} \hat{\mathbf{J}} \hat{\mathbf{P}}_+^* \quad (3.10)$$

acts in the space $L^2(\mathbb{R}_+)$ by the standard formula

$$(\hat{\mathbf{H}}\hat{\mathbf{f}})(t) = (2\pi)^{-1/2} \int_0^\infty \hat{\psi}(t+s) \hat{\mathbf{f}}(s) ds. \quad (3.11)$$

In general, for $\psi \in L^\infty(\mathbb{R})$, formulas (3.9) and (3.11) should of course be understood in the sense of distributions. The function ψ is known as the symbol of the Hankel operator $\hat{\mathbf{H}}$.

3.3. Finally, we note that the representations in the spaces ℓ_+^2 and $L^2(\mathbb{R}_+)$ are connected by the operator $\mathcal{L} = \Phi \mathcal{U} \mathcal{F}^*$. It can be directly expressed in terms of the Laguerre functions, but we do not need the corresponding formulas in this paper.

The relations between different representations can be summarized by the following diagrams:

$$\begin{array}{ccc}
L^2(\mathbb{T}) & \xrightarrow{\mathcal{U}} & L^2(\mathbb{R}; d\nu) \\
\downarrow \mathcal{F} & & \downarrow \Phi \\
\ell^2 & \xrightarrow{\mathcal{L}} & L^2(\mathbb{R}; dt)
\end{array}
\quad
\begin{array}{ccc}
\mathbb{H}_+^2(\mathbb{T}) & \xrightarrow{\mathcal{U}} & \mathbb{H}_+^2(\mathbb{R}) \\
\downarrow \mathcal{F} & & \downarrow \Phi \\
\ell_+^2 & \xrightarrow{\mathcal{L}} & L^2(\mathbb{R}_+)
\end{array}$$

$$\begin{array}{ccc}
f(\mu) & \longrightarrow & \mathbf{f}(\nu) = (\mathcal{U}f)(\nu) \\
\downarrow & & \downarrow \\
\hat{f}_n = (\mathcal{F}f)_n & \longrightarrow & \hat{\mathbf{f}}(t) = (\Phi\mathbf{f})(t)
\end{array}$$

and

$$\begin{array}{ccc}
H & \longrightarrow & \mathbf{H} = \mathcal{U}H\mathcal{U}^* \\
\downarrow & & \downarrow \\
\hat{H} = \mathcal{F}H\mathcal{F}^* & \longrightarrow & \hat{\mathbf{H}} = \Phi\mathbf{H}\Phi^*
\end{array}
\quad
\begin{array}{ccc}
\omega(\mu) & \longrightarrow & \psi(\nu) \\
\downarrow & & \downarrow \\
\hat{\omega}_n & \longrightarrow & \hat{\psi}(t)
\end{array}$$

A Hankel operator H is self-adjoint in $\mathbb{H}_+^2(\mathbb{T})$ if $J\Omega^* = \Omega J$, i.e.,

$$\omega(\bar{\mu}) = \overline{\omega(\mu)}. \quad (3.12)$$

This equality transforms into relations $\overline{\hat{\omega}_n} = \hat{\omega}_n$, $\psi(-\nu) = \overline{\psi(\nu)}$ and $\overline{\hat{\psi}(t)} = \hat{\psi}(t)$ in the spaces ℓ_+^2 , $\mathbb{H}_+^2(\mathbb{R})$ and $L^2(\mathbb{R}_+)$, respectively.

All these definitions can be naturally extended to operators (3.1) acting on functions taking values in an auxiliary Hilbert space \mathcal{N} . For example, formula (3.1) remains meaningful for an operator $H : \mathbb{H}_+^2(\mathbb{T}) \otimes \mathcal{N} \rightarrow \mathbb{H}_+^2(\mathbb{T}) \otimes \mathcal{N}$ if the operator Ω is defined by equality (3.2) where $\omega(\mu) : \mathcal{N} \rightarrow \mathcal{N}$ is an operator-valued function.

3.4. We systematically use the following elementary trick. Instead of the operator (3.1) in the Hardy space $\mathbb{H}_+^2(\mathbb{T})$ we consider the operator $P_+\Omega JP_+$ acting in the space $L^2(\mathbb{T})$. This operator has the same spectrum as the operator (3.1) except for the additional zero eigenvalue of infinite multiplicity. We usually use the same notation H for both of these operators. In particular, this trick allows us to use freely the results of [15]. All results are stated for Hankel operators in $\mathbb{H}_+^2(\mathbb{T})$ while all proofs are given in the space $L^2(\mathbb{T})$.

Let us now come back to the results about the modulus of $|H|$ of Hankel operators $H = P_+\Omega JP_+$ stated in subs. 1.1. We proceed from the following result of [15].

Theorem 3.1. *Suppose that a function $\omega : \mathbb{T} \rightarrow \mathbb{C}$ is continuous apart from some jump discontinuities at finitely many points $a_j \in \mathbb{T}$ with jumps (1.1). At every point of discontinuity $a_j \in \mathbb{T}$, assume condition (1.3). Then the a.c. spectrum of the operator*

$$H_{\text{sym}} = P_+\Omega P_- + P_-\Omega^* P_+ \quad (3.13)$$

acting in the space $L^2(\mathbb{T})$ is given by the relation

$$\text{spec}_{\text{ac}}(H_{\text{sym}}) = \bigcup_j [-2^{-1}|\kappa(a_j)|, 2^{-1}|\kappa(a_j)|].$$

Furthermore, the singular continuous spectrum of H_{sym} is empty and its eigenvalues different from 0 and the points $\pm 2^{-1}|\kappa(a_j)|$ have finite multiplicities and may accumulate only to these points.

It is easy to see that Theorem 3.1 implies the results about the operator $|H|$. Indeed, it follows from (3.13) that

$$\begin{aligned} H_{\text{sym}}^2 &= P_+ \Omega P_- \Omega^* P_+ + P_- \Omega^* P_+ \Omega P_- = P_+ \Omega J P_+ J \Omega^* P_+ + J P_+ J \Omega^* P_+ \Omega J P_+ J \\ &= H H^* + J H^* H J. \end{aligned} \quad (3.14)$$

According to Theorem 3.1 the a.c. spectrum of the operator H_{sym}^2 consists of the union of the intervals $[0, 4^{-1}|\kappa(a_j)|^2]$ (with every interval counted twice). The singular continuous spectrum of H_{sym}^2 is empty and its eigenvalues different from 0 and the points $\pm 4^{-1}|\kappa(a_j)|^2$ have finite multiplicities and may accumulate only to these points. Let us further use that the non-zero parts of the operators $H H^*$ and $J H^* H J$ are unitarily equivalent and that they act in the orthogonal subspaces $\mathbb{H}_+^2(\mathbb{T})$ and $\mathbb{H}_-^2(\mathbb{T})$, respectively. Therefore the results about the operator $|H|$ stated in subs. 1.1 follow from the identity (3.14).

4. MODEL OPERATORS FOR JUMPS AT REAL POINTS

Here we construct “model” operators H_+ , H_- corresponding to jumps of the symbol at the points 1, -1 . The operators H_{\pm} will be directly diagonalized (see Theorem 4.4) with the help of the results on the Mehler operator discussed in subs. 4.1. Then we find (see Theorem 4.5) a class of operators smooth with respect to H_+ and H_- .

4.1. Following [15] as an “elementary model” operator, we choose the Mehler operator defined in the space $L^2(\mathbb{R}_+)$ by the formula

$$(\mathcal{M}u)(t) = \pi^{-1} \int_0^\infty (2+t+s)^{-1} u(s) ds. \quad (4.1)$$

The spectral decomposition of \mathcal{M} is well known and is based on Mehler’s formula:

$$\int_0^\infty \frac{P_{-\frac{1}{2}+i\tau}(1+s)}{2+t+s} ds = \frac{\pi}{\cosh(\pi\tau)} P_{-\frac{1}{2}+i\tau}(1+t), \quad t, \tau \in \mathbb{R}_+, \quad (4.2)$$

where $P_{-\frac{1}{2}+i\tau}(x)$ is the Legendre function. It can be defined (see formulas (2.10.2) and (2.10.5) in the book [1]) for all $x > 1$ in terms of the hypergeometric function $F(a, b, c; z)$ and the gamma-function $\Gamma(\cdot)$ as

$$P_{-\frac{1}{2}+i\tau}(x) = \text{Re} \left(\frac{\Gamma(i\tau)}{\sqrt{\pi}\Gamma(\frac{1}{2}+i\tau)} 2^{\frac{1}{2}+i\tau} F\left(\frac{1}{4} - i\frac{\tau}{2}, \frac{3}{4} - i\frac{\tau}{2}; 1 - i\tau; x^{-2}\right) x^{-\frac{1}{2}+i\tau} \right).$$

It follows that $P_{-\frac{1}{2}+i\tau}(x)$ is a smooth function of $x > 1$ and it has the asymptotics

$$P_{-\frac{1}{2}+i\tau}(x) = \operatorname{Re} \left(\frac{\Gamma(i\tau)}{\sqrt{\pi}\Gamma(\frac{1}{2}+i\tau)} 2^{\frac{1}{2}+i\tau} x^{-\frac{1}{2}+i\tau} \right) + O(x^{-5/2}), \quad x \rightarrow \infty, \quad (4.3)$$

which is differentiable in x and τ . Moreover, the functions $P_{-\frac{1}{2}+i\tau}(x)$ and $P'_{-\frac{1}{2}+i\tau}(x)$ are bounded as $x \rightarrow 1$.

The Mehler-Fock transform Ψ is defined by the formula

$$(\Psi f)(\tau) = \sqrt{\tau \tanh(\pi\tau)} \int_0^\infty P_{-\frac{1}{2}+i\tau}(t+1) f(t) dt \quad (4.4)$$

where $f \in C_0^\infty(\mathbb{R}_+)$ (see, e.g., §3.14 of [1]). Then formula (4.2) can be written as

$$(\Psi \mathcal{M} f)(\tau) = \frac{1}{\cosh(\pi\tau)} (\Psi f)(\tau), \quad \tau > 0. \quad (4.5)$$

A detailed proof of the following assertion can be found in [18].

Lemma 4.1. *Let the operator Ψ be defined by formula (4.4). Then Ψ is a unitary operator in $L^2(\mathbb{R}_+)$ and formula (4.5) holds. In particular, the Mehler operator \mathcal{M} has the purely a.c. spectrum $[0, 1]$ of multiplicity one.*

Remark 4.2. Instead of the Mehler operator, for similar purposes, J. S. Howland used the Hankel operator with kernel $\mathbf{h}(t) = \pi^{-1}e^{-t}t^{-1}$ in [4]. This operator was diagonalized by W. Magnus and M. Rosenblum in [7, 16]. It also has the simple purely a.c. spectrum coinciding with the interval $[0, 1]$, but now the kernel is singular at the point $t = 0$, and the corresponding symbol $\psi(\nu)$ has a jump at infinity.

4.2. The Mehler operator is of course a Hankel operator and, as is well known, its symbol can be chosen as a smooth function with one jump discontinuity. In order to define this symbol, consider the function

$$\zeta(\nu) = \frac{1}{\pi} \int_0^\infty \frac{\sin(\nu t)}{2+t} dt, \quad \nu \in \mathbb{R}. \quad (4.6)$$

Obviously, this function is real and odd. Since

$$\zeta(\nu) = \frac{1}{\pi} \operatorname{Im} \left(e^{-2i\nu} \int_\nu^\infty \frac{e^{2ix}}{x} dx \right), \quad \nu > 0,$$

the function $\zeta \in C^\infty(\mathbb{R} \setminus \{0\})$ and $\zeta(\nu)$ admits an asymptotic expansion in powers ν^{-2k-1} , $k = 0, 1, \dots$ as $\nu \rightarrow \infty$. Moreover, the limits $\zeta(\pm 0)$ exist, $\zeta(\pm 0) = \pm 1/2$ and $\zeta'(\nu) = O(|\ln |\nu||)$ as $\nu \rightarrow 0$. Calculating the Fourier transform of function (4.6), we find that

$$\hat{\zeta}(t) = \frac{-i}{\sqrt{2\pi}} \frac{\operatorname{sign} t}{2+|t|}, \quad t \in \mathbb{R}.$$

Thus the symbol of the operator $\Phi^* \mathcal{M} \Phi$ equals $2i\zeta(\nu)$ and hence

$$\mathcal{M} = \Phi \mathbf{H}(2i\zeta) \Phi^*. \quad (4.7)$$

These results can of course be transplanted onto the unit circle. Let us set

$$v(\mu) = -2i\mu^{-1}\zeta\left(\frac{i}{2}\frac{1+\mu}{1-\mu}\right), \quad (4.8)$$

and let $H(v)$ be the Hankel operator on $\mathbb{H}_+^2(\mathbb{T})$ with this symbol. Note that $v \in C^\infty(\mathbb{T} \setminus \{-1\})$ and the limits $v(-1 \mp i0) = \pm i$ exist so that the jump of v at the point -1 equals

$$v(-1 - i0) - v(-1 + i0) = 2i.$$

Comparing formulas (3.7) and (4.8), we see that the symbol of the operator $\mathcal{U}H(v)\mathcal{U}^*$ also equals $2i\zeta(\nu)$. Hence according to (4.7) we have

$$\mathcal{M} = \Phi\mathcal{U}H(v)\mathcal{U}^*\Phi^*. \quad (4.9)$$

Putting together equalities (4.5) and (4.9), we arrive at the following assertion.

Lemma 4.3. *Let the symbol $v(\mu)$ be defined by formulas (4.6) and (4.8). Then*

$$(FH(v)f)(\tau) = \frac{1}{\cosh(\pi\tau)}(Ff)(\tau), \quad \tau > 0,$$

where $f \in \mathbb{H}_+^2(\mathbb{T})$ is arbitrary and

$$F = \Psi\Phi\mathcal{U} : \mathbb{H}_+^2(\mathbb{T}) \rightarrow L^2(\mathbb{R}_+) \quad (4.10)$$

is the unitary operator.

Thus the operator $H(v)$ reduces to the operator of multiplication by the function $(\cosh(\pi\tau))^{-1}$ in the space $L^2(\mathbb{R}_+)$. Making additionally the change of variables $\lambda = (\cosh(\pi\tau))^{-1}$, we can further reduce the operator $H(v)$ to the operator of multiplication by the independent variable λ in the space $L^2(0, 1)$. However, diagonalization (4.10) is quite convenient for our purposes.

Using the operator $H(v)$ whose symbol (4.8) is singular at the point $\mu = -1$, it is easy to construct a model operator for the singularity at the point $\mu = 1$. Let us set

$$v_+(\mu) = v(-\mu), \quad v_-(\mu) = v(\mu); \quad (4.11)$$

the functions $v_\pm(\mu)$ have the jump $2i$ at the points $\mu = \pm 1$. Note that $H(v_+) = RH(v)\mathcal{R}^*$ where \mathcal{R} is the reflection operator in $\mathbb{H}_+^2(\mathbb{T})$ defined by the formula

$$(\mathcal{R}f)(\mu) = f(-\mu). \quad (4.12)$$

Set

$$F_+ = F\mathcal{R}, \quad F_- = F. \quad (4.13)$$

It follows from Lemma 4.3 that the operator $F_+H(v_+)F_+^*$ acts in the space $L^2(\mathbb{R}_+)$ as multiplication by the function $(\cosh(\pi\tau))^{-1}$. In particular, we obtain the following assertion.

Theorem 4.4. *Let the symbols v_\pm be defined by equalities (4.6), (4.8) and (4.11). The operators $H(v_\pm)$ have the purely a.c. simple spectrum coinciding with the interval $[0, 1]$.*

4.3. Define the operators F_{\pm} by formulas (4.10) and (4.13). In this subsection we consider $H(v_{\pm})$ and F_{\pm} as the operators in $L^2(\mathbb{T})$ extending them by zero onto $\mathbb{H}_+^2(\mathbb{T})^{\perp}$. Let us construct some operators that are smooth (see Definition 2.1 where now $\mathcal{N} = \mathbb{C}$) with respect to $H(v_{\pm})$.

For $a \in \mathbb{T}$, we introduce a function on \mathbb{T} by the equations

$$q_a(\mu) = |\ln |\mu - a||^{-1} \quad \text{for } |\mu - a| \leq e^{-1} \quad (4.14)$$

and $q_a(\mu) = 1$ for $|\mu - a| \geq e^{-1}$. Note that $q_a \in L^{\infty}(\mathbb{T})$ and $q_a(\mu)$ vanishes (logarithmically) only at one point $a \in \mathbb{T}$. Let the operator Q_a in $L^2(\mathbb{T})$ be defined by the formula

$$(Q_a f)(\mu) = q_a(\mu) f(\mu). \quad (4.15)$$

Our goal now is to check the following result.

Theorem 4.5. *Let the symbols v_{\pm} be defined by equalities (4.6), (4.8) and (4.11), and let the operators $Q_{\pm 1}$ be defined by formula (4.15). Then the operator $Q_{\pm 1}^{\beta}$ for $\beta > 1/2$ is strongly $H(v_{\pm})$ -smooth on the interval $(0, 1)$ for the diagonalization F_{\pm} with any exponent $\gamma < \beta - 1/2$.*

Let us start with an informal interpretation of the result of Theorem 4.5. By formula (4.2) up to a normalization, eigenfunctions (of the continuous spectrum) of the operator \mathcal{M} equal $\vartheta_{\tau}(t) = P_{-\frac{1}{2}+i\tau}(1+t)$. In view of (4.3), they do not belong to L^2 at infinity. This implies that the eigenfunctions $(\Phi^* \vartheta_{\tau})(\nu)$ of the operator $\Phi^* \mathcal{M} \Phi$ do not belong to L^2 in a neighbourhood of the point $\nu = 0$ and hence eigenfunctions $(\mathcal{U}^* \Phi^* \vartheta_{\tau})(\mu)$ of the operator $H(v)$ do not belong to L^2 in a neighbourhood of the point $\mu = -1$. However Lemma 4.6 below shows that the singularities of the functions $(\mathcal{U}^* \Phi^* \vartheta_{\tau})(\mu)$ at the point $\mu = -1$ are rather weak. This is an indication that an operator of multiplication by a bounded function is $H(v)$ -smooth provided this function logarithmically vanishes at the point $\mu = -1$. We note that the symbol $v(\mu)$ has a jump at the point $\mu = -1$.

The formal proof requires some elementary information on the Fourier transform of the Legendre functions.

Lemma 4.6. *The integral*

$$w_{\tau}(\nu) = (2\pi)^{-1/2} \int_1^{\infty} P_{-\frac{1}{2}+i\tau}(x) e^{-i\nu x} dx, \quad \tau > 0, \quad (4.16)$$

converges for all $\nu > 0$, and it is differentiable in τ . If $\Delta \subset \mathbb{R}_+$ is a compact interval and $\tau \in \Delta$, then there exists $C = C(\Delta)$ such that

$$|w_{\tau}(\nu)| \leq C|\nu|^{-1}, \quad |\partial w_{\tau}(\nu)/\partial \tau| \leq C|\nu|^{-1}, \quad |\nu| \geq 1/2, \quad (4.17)$$

and

$$|w_{\tau}(\nu)| \leq C|\nu|^{-1/2}, \quad |\partial w_{\tau}(\nu)/\partial \tau| \leq C|\nu|^{-1/2} |\ln |\nu||, \quad |\nu| \leq 1/2, \quad \nu \neq 0. \quad (4.18)$$

Moreover, the integral

$$\int_1^R P_{-\frac{1}{2}+i\tau}(x)e^{-i\nu x}dx \quad (4.19)$$

is bounded by $C|\nu|^{-1}$ for $|\nu| \geq 1/2$ and by $C|\nu|^{-1/2}$ for $|\nu| \leq 1/2$ with a constant C that does not depend on $R \leq \infty$.

Estimates (4.17) and (4.18) are proven in [15]; see Lemma 3.10. The assertion about the integral (4.19) can be obtained in exactly the same way.

Next, we derive a convenient representation for the operator $\Psi\Phi$. According to our convention of subs. 3.4 we put $\Psi\mathbf{f} = 0$ for $\mathbf{f} \in L^2(\mathbb{R}_-)$ and consider $\Psi\Phi$ as a mapping of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R}_+)$. Denote by $\mathcal{S} = \mathcal{S}(\mathbb{R})$ the Schwartz space.

Lemma 4.7. *Let $\mathbf{g} \in \mathcal{S}$. Then*

$$(\Psi\Phi\mathbf{g})(\tau) = \sqrt{\tau \tanh(\pi\tau)}\psi(\tau) \quad (4.20)$$

where

$$\psi(\tau) = \int_{-\infty}^{\infty} \mathbf{g}(\nu)w_{\tau}(\nu)e^{i\nu\tau}d\nu \quad (4.21)$$

and $w_{\tau}(\nu)$ is function (4.16).

Proof. It follows from (4.4) that

$$(\Psi\Phi\mathbf{g})(\tau) = (2\pi)^{-1/2}\sqrt{\tau \tanh(\pi\tau)} \lim_{R \rightarrow \infty} \int_0^R dt P_{-\frac{1}{2}+i\tau}(t+1) \left(\int_{-\infty}^{\infty} \mathbf{g}(\nu)e^{-i\nu t}d\nu \right).$$

Changing the order of integrations by the Fubini theorem, we obtain representation (4.20) with

$$\psi(\tau) = (2\pi)^{-1/2} \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} \mathbf{g}(\nu) \left(\int_0^R P_{-\frac{1}{2}+i\tau}(t+1)e^{-i\nu t}dt \right) d\nu.$$

Using the assertion of Lemma 4.6 about integral (4.19), we can pass here to the limit by the dominated convergence theorem. \square

Now we are in a position to complete the proof of Theorem 4.5.

Proof. Consider, for example, the sign “ $-$ ”. We have to check the estimates

$$|(FQ_{-1}^{\beta}f)(\tau)| \leq C\|f\|_{L^2(\mathbb{T})} \quad (4.22)$$

and

$$|(FQ_{-1}^{\beta}f)(\tau') - (FQ_{-1}^{\beta}f)(\tau)| \leq C|\tau' - \tau|^{\gamma}\|f\|_{L^2(\mathbb{T})}, \quad \gamma < \beta - 1/2, \quad (4.23)$$

for τ and τ' in compact subintervals of \mathbb{R}_+ and all $f \in L^2(\mathbb{T})$. We can of course assume that f belongs to the set $\mathcal{U}^*\mathcal{S}$ dense in $L^2(\mathbb{T})$.

Put $g = Q_{-1}^\beta f$, $\mathbf{f} = \mathcal{U}f$, $\mathbf{g} = \mathcal{U}g$ and $\mathbf{q}(\nu) = |\ln |\nu||^{-1}$ for $|\nu| \leq e^{-1}$, $\mathbf{q}(\nu) = 1$ for $|\nu| \geq e^{-1}$. Then using notation (4.20), we can equivalently rewrite estimates (4.22) and (4.23) as

$$|\psi(\tau)| \leq C \|\mathbf{q}^{-\beta} \mathbf{g}\| \quad (4.24)$$

and

$$|\psi(\tau') - \psi(\tau)| \leq C |\tau' - \tau|^\gamma \|\mathbf{q}^{-\beta} \mathbf{g}\|, \quad \gamma < \beta - 1/2. \quad (4.25)$$

By the Schwarz inequality, it follows from representation (4.21) that

$$|\psi(\tau)| \leq \|\mathbf{q}^\beta w_\tau\| \|\mathbf{q}^{-\beta} \mathbf{g}\|$$

where $\mathbf{q}^\beta w_\tau \in L^2(\mathbb{R})$ if $\beta > 1/2$ by virtue of Lemma 4.6. This proves (4.24). Estimate (4.25) can be obtained quite similarly if one takes into account that $|w_{\tau'}(\nu) - w_\tau(\nu)|$ is bounded by $C|\tau' - \tau||\nu|^{-1}$ for $|\nu| \geq 1/2$ and by $C|\tau' - \tau|^\gamma |\nu|^{-1/2} |\ln |\nu||^\gamma$ for $|\nu| \leq 1/2$ according to estimates (4.17) and (4.18), respectively. \square

5. MODEL OPERATORS FOR JUMPS AT COMPLEX POINTS

Here we construct “model” operators H_a for pairs a, \bar{a} of complex conjugate points. Although the operators H_a are not Hankel in the space $\mathbb{H}_+^2(\mathbb{T})$, they can be realized as Hankel operators in the space of two components vector valued functions. In the first subsection, we describe the necessary representation of $\mathbb{H}_+^2(\mathbb{T})$. The operators H_a are diagonalized in Theorem 5.3 and the class of H_a -smooth operators is found in Theorem 5.4.

5.1. Let us identify the spaces $L^2(\mathbb{T}) \otimes \mathbb{C}^2$ and $L^2(\mathbb{T})$ and, in particular, their subspaces $\mathbb{H}_+^2(\mathbb{T}) \otimes \mathbb{C}^2$ and $\mathbb{H}_+^2(\mathbb{T})$. Put $f_{\text{even}} = 2^{-1}(I + R)f$, $f_{\text{odd}} = 2^{-1}(I - R)f$ where R is operator (4.12). Evidently, f_{even} and f_{odd} are the even and odd parts of f . Using notation (3.3), (3.4), we set

$$\begin{aligned} f^{(+)}(\mu) &:= f_{\text{even}}(\mu^{1/2}) = \sum_{n=-\infty}^{\infty} \hat{f}_{2n} \mu^n, \\ f^{(-)}(\mu) &:= \mu^{-1/2} f_{\text{odd}}(\mu^{1/2}) = \sum_{n=-\infty}^{\infty} \hat{f}_{2n+1} \mu^n. \end{aligned} \quad (5.1)$$

Then²

$$\mathbf{U} : f(\mu) \mapsto (f^{(+)}(\mu), f^{(-)}(\mu))^\top =: \vec{f}(\mu) \quad (5.2)$$

is a unitary mapping of $L^2(\mathbb{T})$ onto $L^2(\mathbb{T}) \otimes \mathbb{C}^2$ and of $\mathbb{H}_+^2(\mathbb{T})$ onto $\mathbb{H}_+^2(\mathbb{T}) \otimes \mathbb{C}^2$. Obviously, the operator $\mathbf{U}^* : L^2(\mathbb{T}) \otimes \mathbb{C}^2 \rightarrow L^2(\mathbb{T})$ acts by the formula

$$(\mathbf{U}^* \vec{f})(\mu) = f^{(+)}(\mu^2) + \mu f^{(-)}(\mu^2).$$

²The upper index “ \top ” means that a vector is regarded as a column.

A Hankel operator $H = H(\Sigma)$ in the space $\mathbb{H}_+^2(\mathbb{T}) \otimes \mathbb{C}^2$ is defined by formula (3.1) where $(J\vec{f})(\mu) = \bar{\mu}\vec{f}(\bar{\mu})$, $(\Omega\vec{f})(\mu) = \mu\Sigma(\mu)\vec{f}(\mu)$ and the symbol

$$\Sigma(\mu) = \begin{pmatrix} \sigma_{1,1}(\mu) & \sigma_{1,2}(\mu) \\ \sigma_{2,1}(\mu) & \sigma_{2,2}(\mu) \end{pmatrix}. \quad (5.3)$$

is a 2×2 matrix-valued function.

An easy calculation shows that, for a Hankel operator $H(\omega) : \mathbb{H}_+^2(\mathbb{T}) \rightarrow \mathbb{H}_+^2(\mathbb{T})$, the operator

$$H(\Sigma) := UH(\omega)U^* : \mathbb{H}_+^2(\mathbb{T}) \otimes \mathbb{C}^2 \rightarrow \mathbb{H}_+^2(\mathbb{T}) \otimes \mathbb{C}^2$$

is also the Hankel operator with symbol (5.3) where

$$\begin{aligned} \sigma_{1,1}(\mu) &= \omega_{\text{even}}(\mu^{1/2}), & \sigma_{2,2}(\mu) &= \mu^{-1}\omega_{\text{even}}(\mu^{1/2}), \\ \sigma_{1,2}(\mu) &= \sigma_{2,1}(\mu) = \mu^{-1/2}\omega_{\text{odd}}(\mu^{1/2}). \end{aligned} \quad (5.4)$$

On the other hand, for a Hankel operator $H(\Sigma)$ in the space $\mathbb{H}_+^2(\mathbb{T}) \otimes \mathbb{C}^2$, the operator $U^*H(\Sigma)U =: H(\Sigma)$ is Hankel in the space $\mathbb{H}_+^2(\mathbb{T})$ if and only if $\sigma_{1,1}(\mu) = \mu\sigma_{2,2}(\mu)$ and $\sigma_{1,2}(\mu) = \sigma_{2,1}(\mu)$. In this case the symbol $\omega(\mu)$ of this operator can be constructed by formulas (5.4).

5.2. First, we construct a model operator corresponding to the pair $(i, -i)$. Clearly, the symbol

$$\omega_\varphi(\mu) = (\sin \varphi - \mu \cos \varphi)v(\mu^2), \quad \varphi \in [0, 2\pi), \quad (5.5)$$

is smooth everywhere except the points $\pm i$ where it has the jumps $\pm 2e^{\pm i\varphi}$. Although the function $\omega_\varphi(\mu)$ looks simple, we do not know how to diagonalize the Hankel operator $H(\omega_\varphi)$ explicitly. Therefore we distinguish its singular part which will be done in the representation $\mathbb{H}_+^2(\mathbb{T}) \otimes \mathbb{C}^2$. It follows from formulas (5.3) and (5.4) that the symbol of the Hankel operator $UH(\omega_\varphi)U^* =: H(\Sigma_\varphi)$ acting in the space $\mathbb{H}_+^2(\mathbb{T}) \otimes \mathbb{C}^2$ is the matrix-valued function

$$\Sigma_\varphi(\mu) = \Sigma_\varphi^{(0)}(\mu) + \tilde{\Sigma}_\varphi(\mu) \quad (5.6)$$

where

$$\Sigma_\varphi^{(0)}(\mu) = \begin{pmatrix} \sin \varphi & -\cos \varphi \\ -\cos \varphi & -\sin \varphi \end{pmatrix} v(\mu) \quad (5.7)$$

and

$$\tilde{\Sigma}_\varphi(\mu) = \begin{pmatrix} 0 & 0 \\ 0 & \sin \varphi \end{pmatrix} (1 + \bar{\mu})v(\mu). \quad (5.8)$$

Note that the symbol $\Sigma_\varphi^{(0)}(\mu)$ has a jump at the point $\mu = -1$ while the symbol $\tilde{\Sigma}_\varphi(\mu)$ is a Lipschitz continuous function. Equality (5.6) implies that

$$H(\omega_\varphi) = U^*H(\Sigma_\varphi^{(0)})U + U^*H(\tilde{\Sigma}_\varphi)U. \quad (5.9)$$

Diagonalizing the 2×2 matrix in the right-hand side of (5.7), we see that

$$\Sigma_\varphi^{(0)}(\mu) = v(\mu)Y_\varphi^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Y_\varphi$$

where

$$Y_\varphi = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\cos \varphi}{\sqrt{1-\sin \varphi}} & -\sqrt{1-\sin \varphi} \\ \frac{\cos \varphi}{\sqrt{1+\sin \varphi}} & \sqrt{1+\sin \varphi} \end{pmatrix}. \quad (5.10)$$

It follows that

$$\mathbf{H}(\Sigma_\varphi^{(0)}) = Y_\varphi^* \begin{pmatrix} H(v) & 0 \\ 0 & -H(v) \end{pmatrix} Y_\varphi.$$

Thus Lemma 4.3 yields the explicit diagonalization of the operator $\mathbf{H}(\Sigma_\varphi^{(0)})$. In particular, the following result is a direct consequence of Theorem 4.4.

Lemma 5.1. *The operators $\mathbf{H}(\Sigma_\varphi^{(0)})$ and hence $H(\Sigma_\varphi^{(0)}) := \mathbf{U}^* \mathbf{H}(\Sigma_\varphi^{(0)}) \mathbf{U}$ have the purely a.c. simple spectrum coinciding with the interval $[-1, 1]$.*

Note that in the particular case $\sin \varphi = 0$, we have $\Sigma_\varphi(\mu) = \Sigma_\varphi^{(0)}(\mu)$ so that the operator $H(\omega_\varphi)$ can be explicitly diagonalized.

5.3. The case of jumps at arbitrary pairs (a, \bar{a}) , $\operatorname{Im} a > 0$, of complex conjugate points of \mathbb{T} can be reduced to the case $a = i$. To that end, we consider the transformation of symbols under special fractional linear maps of the unit disc corresponding to dilations of the upper half-plane.

Lemma 5.2. *Put*

$$(T_\alpha f)(\mu) = \sqrt{1-\alpha^2} (1+\alpha\mu)^{-1} f\left(\frac{\mu+\alpha}{1+\alpha\mu}\right), \quad \alpha \in (-1, 1). \quad (5.11)$$

Then T_α is the unitary operator in $\mathbb{H}_+^2(\mathbb{T})$ and, for an arbitrary Hankel operator $H(\omega)$, we have

$$T_\alpha^* H(\omega) T_\alpha = H(\omega^{(\alpha)}) \quad (5.12)$$

where

$$\omega^{(\alpha)}(\mu) = \mu^{-1} \frac{\mu - \alpha}{1 - \alpha\mu} \omega\left(\frac{\mu - \alpha}{1 - \alpha\mu}\right). \quad (5.13)$$

Proof. It is convenient to make calculations in the representation $\mathbb{H}_+^2(\mathbb{R})$. It follows from formulas (3.5) and (3.6) that the operator $\mathbf{T}_\alpha = \mathcal{U} T_\alpha \mathcal{U}^*$ is the dilation:

$$(\mathbf{T}_\alpha \mathbf{f})(\nu) = \sqrt{\frac{1+\alpha}{1-\alpha}} \mathbf{f}\left(\frac{1+\alpha}{1-\alpha} \nu\right).$$

Therefore for a Hankel operator $\mathbf{H}(\psi)$ with symbol $\psi(\nu)$, we have

$$\mathbf{T}_\alpha^* \mathbf{H}(\psi) \mathbf{T}_\alpha = \mathbf{H}(\psi_\alpha) \quad (5.14)$$

where

$$\psi^{(\alpha)}(\nu) = \psi\left(\frac{1-\alpha}{1+\alpha} \nu\right). \quad (5.15)$$

Recall that the symbols of the operators $\mathbf{H}(\psi) = \mathcal{U} H(\omega) \mathcal{U}^*$ and $H(\omega)$ are linked by formula (3.7). Hence using (5.14), (5.15) and making pull back to $\mathbb{H}_+^2(\mathbb{T})$, we obtain formulas (5.12), (5.13). \square

For a pair (a, \bar{a}) where $a = e^{i\theta}$, $\theta \in (0, \pi)$, we now put

$$\omega_{\varphi, \theta}(\mu) = \mu^{-1} \frac{\mu - \alpha}{1 - \alpha\mu} \omega_{\varphi}\left(\frac{\mu - \alpha}{1 - \alpha\mu}\right) \quad (5.16)$$

where $\omega_{\varphi}(\mu)$ is symbol (5.5) and

$$\alpha = \frac{a - i}{1 - ia} = \tan(\pi/4 - \theta/2).$$

Observe that the transformation $\mu \mapsto (\mu - \alpha)(1 - \alpha\mu)^{-1}$ sends the pair (a, \bar{a}) into $(i, -i)$. Therefore the symbol $\omega_{\varphi, \theta}(\mu)$ is smooth everywhere except the points a and \bar{a} where it has the jumps $2ie^{i(\varphi-\theta)}$ and $2ie^{i(\theta-\varphi)}$, respectively. It follows from Lemma 5.2 that

$$H(\omega_{\varphi, \theta}) = T_{\alpha}^* H(\omega_{\varphi}) T_{\alpha} \quad \text{where} \quad \alpha = \tan(\pi/4 - \theta/2).$$

Putting together this equality with (5.9), we find that

$$H(\omega_{\varphi, \theta}) = H_{\varphi, \theta}^{(0)} + \tilde{H}_{\varphi, \theta} \quad (5.17)$$

where

$$H_{\varphi, \theta}^{(0)} = T_{\alpha}^* \mathbf{U}^* \mathbf{H}(\Sigma_{\varphi}^{(0)}) \mathbf{U} T_{\alpha} \quad (5.18)$$

and

$$\tilde{H}_{\varphi, \theta} = T_{\alpha}^* \mathbf{U}^* \mathbf{H}(\tilde{\Sigma}_{\varphi}) \mathbf{U} T_{\alpha}. \quad (5.19)$$

Of course Lemma 5.1 yields the explicit diagonalization of the operator $H_{\varphi, \theta}^{(0)}$.

Let us summarize the results obtained.

Theorem 5.3. *Let the symbol $\omega_{\varphi, \theta}$ be defined by formulas (5.5) and (5.16), and let the symbols $\Sigma_{\varphi}^{(0)}$, $\tilde{\Sigma}_{\varphi}$ be defined by formulas (5.7), (5.8). Then:*

1⁰ *Equalities (5.17) – (5.19) hold.*

2⁰ *The operator $H_{\varphi, \theta}^{(0)}$ has the purely a.c. simple spectrum coinciding with the interval $[-1, 1]$.*

5.4. Similarly to subs. 4.3, we now extend all operators from $\mathbb{H}_+^2(\mathbb{T})$ onto $L^2(\mathbb{T})$ setting them to zero on $\mathbb{H}_+^2(\mathbb{T})^{\perp}$. We again use Definition 2.1 with $\mathcal{N} = \mathbb{C}$. Our goal is to check the following result.

Theorem 5.4. *Let the operator $H_{\varphi, \theta}^{(0)}$ be defined by formula (5.18), and let the operator Q_a be defined by formulas (4.14) and (4.15). Then the operator $Q_a^{\beta} Q_{\bar{a}}^{\beta}$ for $\beta > 1/2$ is strongly $H_{\varphi, \theta}^{(0)}$ -smooth on $(-1, 0) \cup (0, 1)$ for the diagonalization $FY_{\varphi} \mathbf{U} T_{\alpha}$ with any exponent $\gamma < \beta - 1/2$.*

Proof. Let us check the first estimate (2.2), that is,

$$|(FY_{\varphi} \mathbf{U} T_{\alpha} Q_a^{\beta} Q_{\bar{a}}^{\beta} f)(\tau)| \leq C \|f\|_{L^2(\mathbb{T})} \quad (5.20)$$

for τ in compact subintervals of \mathbb{R}_+ and all $f \in L^2(\mathbb{T})$. Observe that the operator $T_{\alpha} Q_a^{\beta} Q_{\bar{a}}^{\beta} T_{\alpha}^*$ acts as the multiplication by a function bounded by $C q_i(\mu)^{\beta} q_{-i}(\mu)^{\beta}$.

Therefore the proof reduces to the case $a = i$ when $T_\alpha = I$. Next, we use that the function $q_i(\mu)q_{-i}(\mu)$ is even so that we can set $g(\mu^2) = q_i(\mu)q_{-i}(\mu)$. Let G be the operator of multiplication by $g(\mu)$. By the definition (5.1), (5.2) of the operator \mathbf{U} , we have $\mathbf{U}Q_i^\beta Q_{-i}^\beta f = G^\beta \mathbf{U}f$. Since the operators Y_φ and G^β commute, estimate (5.20) for $a = i$ can be rewritten as

$$|(FG^\beta Y_\varphi \mathbf{U}f)(\tau)| \leq C\|f\|_{L^2(\mathbb{T})}. \quad (5.21)$$

Note that the operator $Y_\varphi \mathbf{U}$ is unitary and that the function $g(\mu)$ is bounded by $C|\ln|\mu+1||^{-1}$ as $\mu \rightarrow -1$. Thus for the proof of (5.21), it remains to use that according to Theorem 4.5 the operator G^β is smooth with respect to $H(v)$ (see estimate (4.22)). The second estimate (2.2) can be verified quite similarly. \square

6. MAIN RESULTS

The main results are stated in subs. 1 and proven in subs. 3. Necessary compactness results are collected in subs. 2. Then we reformulate our results in subs. 4 in the representation $\mathbb{H}_+^2(\mathbb{R})$ of Hankel operators. The case of matrix-valued symbols is discussed in subs. 5.

6.1. Let $\omega \in L^\infty(\mathbb{T})$ be a symbol satisfying condition (3.12), and let $H(\omega)$ be the corresponding self-adjoint Hankel operator (3.1) in $\mathbb{H}_+^2(\mathbb{T})$. Our aim is to perform the spectral analysis of Hankel operators with piecewise continuous symbols $\omega(\mu)$.

For a point $a \in \mathbb{T}$ of the discontinuity of $\omega(\mu)$, we define the jump by formula (1.1) and assume condition (1.3). By (3.12), if $\omega(\mu)$ has a jump \varkappa at some point $a \in \mathbb{T}$, then it also has the jump $-\bar{\varkappa}$ at the point \bar{a} . In particular, the jumps at the points ± 1 are purely imaginary. Let us write the points of discontinuity of ω as $1, -1, a_1, \dots, a_{N_0}, \bar{a}_1, \dots, \bar{a}_{N_0}$, $\text{Im } a_j > 0$, and the jumps of ω at these points as

$$\varkappa(\pm 1) = 2i\kappa_\pm, \quad \kappa_\pm \in \mathbb{R}, \quad \varkappa(a_j) = 2\kappa_j e^{i\psi_j}, \quad \kappa_j > 0. \quad (6.1)$$

Of course, the points 1 or -1 may be regular; in this case condition (1.3) at these points is not required and $\kappa_+ = 0$ or $\kappa_- = 0$.

Thus, we accept

Assumption 6.1. *A function $\omega : \mathbb{T} \rightarrow \mathbb{C}$ satisfies the self-adjointness condition (3.12) and is continuous apart from some jump discontinuities at finitely many points $1, -1, a_1, \dots, a_{N_0}, \bar{a}_1, \dots, \bar{a}_{N_0}$ with jumps (6.1). At every point of discontinuity $a \in \mathbb{T}$, it satisfies condition (1.3).*

Recall that \mathbf{A}_Δ is the operator of multiplication by independent variable in the space $L^2(\Delta)$. We put

$$\Delta_\pm = [0, \kappa_\pm] \quad \text{and} \quad \Delta_j = [-\kappa_j, \kappa_j]. \quad (6.2)$$

The spectral structure of the operator $H = H(\omega)$ is described in the following assertion.

Theorem 6.2. *Let Assumption 6.1 hold and $H = H(\omega)$. Then:*

1⁰ *If $\beta_0 > 1$, then the operator $H^{(\text{ac})}$ is unitarily equivalent to the orthogonal sum*

$$A_{\Delta_+} \oplus A_{\Delta_-} \oplus \bigoplus_{j=1}^{N_0} A_{\Delta_j}. \quad (6.3)$$

2⁰ *If $\beta_0 > 2$, then the singular continuous spectrum of the operator H is empty and its eigenvalues, distinct from 0, κ_{\pm} and $\pm\kappa_j$, $j = 1, \dots, N_0$, have finite multiplicities and can accumulate only to these points.*

Recall that, to a given Hankel operator, there correspond various symbols whose differences belong to the space $H_-^\infty(\mathbb{T})$. Nevertheless the formulation of Theorem 6.2 has an intrinsic nature because functions in $H_-^\infty(\mathbb{T})$ cannot have jumps.

Next, we state the limiting absorption principle for the operator H .

Theorem 6.3. *Let Assumption 6.1 hold with $\beta_0 > 2$ and $H = H(\omega)$. Let $Q : H_+^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ be the operator of multiplication by a bounded function $q(\mu)$ such that*

$$q(\mu) = O(|\ln |\mu - a||^{-\beta}), \quad \mu \rightarrow a, \quad \beta > 1,$$

in all points of discontinuity of ω , that is, in $a = 1, -1, a_1, \dots, a_{N_0}, \bar{a}_1, \dots, \bar{a}_{N_0}$. Then the operator-valued function $Q(H - z)^{-1}Q^$ is continuous in z if $\pm \text{Im } z \geq 0$ away from all points 0, κ_{\pm} and $\pm\kappa_j$, $j = 1, \dots, N_0$, and eigenvalues of the operator H .*

Finally, we consider the wave operators. Recall that the symbols v_{\pm} were defined by equalities (4.6), (4.8) and (4.11). The matrix symbol $\Sigma_{\varphi}^{(0)}$ was defined by formula (5.7) and the model operator $H_{\varphi, \theta}^{(0)}$ was defined by formulas (5.2), (5.11) and (5.18).

Theorem 6.4. *Let Assumption 6.1 hold with $\beta_0 > 1$ and $H = H(\omega)$. Let the numbers κ_{\pm} , κ_j and ψ_j be defined by formula (6.1). Put $a_j = e^{i\theta_j}$ and $\varphi_j = \psi_j + \theta_j - \pi/2$. Then all the assertions of Theorem 2.6 are true with $N = N_0 + 2$ for the operators H and $H_j = \kappa_j H_{\varphi_j, \theta_j}^{(0)}$ if $j = 1, \dots, N_0$ and $H_{N_0+1} = \kappa_+ H(v_+)$, $H_{N_0+2} = \kappa_- H(v_-)$.*

In particular cases where a symbol ω has only one real singularity or only one pair of complex singularities, we have the following results.

Corollary 6.5. *Suppose that a symbol $\omega(\mu)$ has only one jump $2i\kappa_+$ at the point 1 or $2i\kappa_-$ at -1 . Then the corresponding wave operators $W_{\tau}(H(\omega), \kappa_+ H(v_+))$ or $W_{\tau}(H(\omega), \kappa_- H(v_-))$ (for both signs “ $\tau = \pm$ ”) exist and are complete.*

Corollary 6.6. *Suppose that a symbol $\omega(\mu)$ has only two jumps $2\kappa e^{i\psi}$ and $-2\kappa e^{-i\psi}$, $\kappa > 0$, at points $e^{i\theta}$ and $e^{-i\theta}$, respectively. Put $\varphi = \psi + \theta - \pi/2$. Then the wave operators $W_{\pm}(H(\omega), \kappa H_{\varphi, \theta}^{(0)})$ exist and are complete.*

Observe that $H(\Sigma_{\varphi, \theta}^{(0)})$ is not (if $\sin \varphi \neq 0$) a Hankel operator in the space $\mathbb{H}_+^2(\mathbb{T})$. It is however possible to reformulate Theorem 6.4 solely in terms of Hankel operators. As usual, the symbols below satisfy condition (3.12).

Theorem 6.7. *Let Assumption 6.1 hold with $\beta_0 > 1$ and $H = H(\omega)$. Let $\varkappa(\pm 1)$, $\varkappa(a_j)$, $\text{Im } a_j > 0$, be the jumps of $\omega(\mu)$. Suppose that symbols ω_{\pm} satisfy Assumption 6.1 and that their only jumps $\varkappa(\pm 1)$ are located at the points ± 1 . Suppose also that symbols ω_j , $j = 1, \dots, N_0$, satisfy Assumption 6.1 and that their only jumps $\varkappa(a_j)$ in the upper half-plane are located at the points a_j . Then all the assertions of Theorem 2.6 with $N = N_0 + 2$ are true for the operators $H_j = H(\omega_j)$, $j = 1, \dots, N_0$, $H_{N_0+1} = H(\omega_+)$, $H_{N_0+2} = H(\omega_-)$.*

Proof. Let the index τ below take both values “+” and “−”. It follows from Corollary 6.5 that the wave operators $W_{\tau}(H(\omega_{\pm}), \kappa_{\pm}H(v_{\pm}))$ exist and are complete. Therefore, by the multiplication theorem for wave operators (see relation (2.3)), the wave operators $W_{\tau}(H, H(\omega_{\pm}))$ also exist and

$$W_{\tau}(H, H(\omega_{\pm})) = W_{\tau}(H, \kappa_{\pm}H(v_{\pm}))W_{\tau}^*(H(\omega_{\pm}), \kappa_{\pm}H(v_{\pm})). \quad (6.4)$$

Similarly, it follows from Corollary 6.6 that the wave operators $W_{\tau}(H(\omega_j), \kappa_j H_{\varphi_j, \theta_j}^{(0)})$ exist and are complete. Therefore, by the multiplication theorem, the wave operators $W_{\tau}(H, H(\omega_j))$ also exist and

$$W_{\tau}(H, H(\omega_j)) = W_{\tau}(H, \kappa_j H_{\varphi_j, \theta_j}^{(0)})W_{\tau}^*(H(\omega_j), \kappa_j H_{\varphi_j, \theta_j}^{(0)}). \quad (6.5)$$

Relations (6.4) and (6.5) imply that

$$R(W_{\tau}(H, H(\omega_{\pm}))) = R(W_{\tau}(H, \kappa_{\pm}H(v_{\pm})))$$

and

$$R(W_{\tau}(H, H(\omega_j))) = R(W_{\tau}(H, \kappa_j H_{\varphi_j, \theta_j}^{(0)})).$$

Therefore in the statement of Theorem 6.4, the wave operators $W_{\tau}(H, \kappa_{\pm}H(v_{\pm}))$ can be replaced by $W_{\tau}(H, H(\omega_{\pm}))$, and the wave operators $W_{\tau}(H, \kappa_j H_{\varphi_j, \theta_j}^{(0)})$ can be replaced by $W_{\tau}(H, H(\omega_j))$. \square

Recall that the functions $\omega_{\varphi, \theta}$ defined by formulas (5.5), (5.16) are smooth away from the points $e^{i\theta}$ and $e^{-i\theta}$ and have the jumps $2ie^{i(\varphi-\theta)}$ and $2ie^{i(\theta-\varphi)}$ at these points. Therefore for the symbol ω_j in Theorem 6.7, we can, for example, choose the function

$$\omega_j(\mu) = \kappa_j \omega_{\varphi_j, \theta_j}(\mu) \quad (6.6)$$

where $a_j = e^{i\theta_j}$, $\varphi_j = \psi_j + \theta_j - \pi/2$. Similarly, since the functions $v_{\pm}(\mu)$ defined by formulas (4.6), (4.8) and (4.11) are smooth away from the points ± 1 where they have the jump $2i$, we can set $\omega_{\pm}(\mu) = \kappa_{\pm} v_{\pm}(\mu)$.

It follows from Theorem 6.7 that, for all $f \in \mathcal{H}^{(\text{ac})}(H)$, the relation

$$e^{-iHt}f = \sum_{j=1}^N e^{-iH_j t} f_j^{(\pm)} + \varepsilon^{(\pm)}(t)$$

holds with $f_j^{(\pm)} = W_{\pm}(H, H_j)^* f$ and $\varepsilon^{(\pm)}(t) \rightarrow 0$ as $t \rightarrow \pm\infty$. In particular, this relation shows that asymptotically the functions $(e^{-iHt}f)(\mu)$ are concentrated for large

$|t|$ in neighborhoods of singular points of the symbol $\omega(\mu)$. In a somewhat more simple situation, this phenomenon was discussed in a detailed way in [19].

6.2. In addition to the results of Sections 4 and 5 on model operators, for the proof of Theorems 6.2, 6.3 and 6.4 we also need the results of [15], Section 4, on the boundedness and compactness of Hankel operators sandwiched by singular weights. These results will be stated in this subsection. We emphasize that now $H(\omega) = P_+ \Omega J P_+$ are considered as operators in the space $L^2(\mathbb{T})$. Condition (3.12) is supposed to be satisfied.

Recall that the function $q_a(\mu)$ is defined by equality (4.14). Let Q be the operator of multiplication by the function

$$q(\mu) = \left(q_1(\mu) q_{-1}(\mu) \prod_{j=1}^N q_{a_j}(\mu) q_{\bar{a}_j}(\mu) \right)^\beta, \quad \beta > 0, \quad (6.7)$$

vanishing in all singular points of ω . Of course $Q = Q^*$ and its kernel is trivial.

The first assertion follows from the classical Muckenhoupt result [9].

Lemma 6.8. *Suppose that $\omega \in L^\infty(\mathbb{T})$. Then the operators $Q P_+ \omega P_- Q^{-1}$ and hence $Q H(\omega) Q^{-1}$ are bounded.*

The next statement generalizes the well known result about the compactness of Hankel operators $H(\omega)$ with $\omega \in C(\mathbb{T})$.

Lemma 6.9. *Let $\omega \in C(\mathbb{T})$, and let condition (1.3) with some $\beta_0 > 2\beta$ be satisfied in all singular points of ω . Then the operators $Q^{-1} P_+ \omega P_- Q^{-1}$ and hence $Q^{-1} H(\omega) Q^{-1}$ are compact.*

Lemma 6.10. *Let the symbols ω_j for $j = 1, \dots, N_0$ be defined by formula (6.6), $\omega_{N_0+1} = \kappa_+ v_+$ and $\omega_{N_0+2} = \kappa_- v_-$. Then, for $n, m = 1, \dots, N_0 + 2$ and $n \neq m$, all operators $Q^{-1} P_+ \omega_n P_- \omega_m P_+ Q^{-1}$ and hence $Q^{-1} H(\omega_n) H(\omega_m) Q^{-1}$ are compact.*

The proof of this result uses that the singularities of the symbols ω_n and ω_m are disjoint if $n \neq m$.

6.3. For the proof of all Theorems 6.2, 6.3 and 6.4, we have to check Assumption 2.3 with the operators H , H_n , $n = 1, \dots, N = N_0 + 2$, defined in Theorem 6.4 and the operator \tilde{H} defined by equality (2.4). For the smooth operator Q , we choose the operator of multiplication by function (6.7) where $\beta \in (1/2, \beta_0/2)$. Since $\beta > 1/2$, Assumption 2.3a is satisfied with any $\gamma < \beta - 1/2$ according to Theorems 4.5 and 5.4.

Let the jumps of $\omega(\mu)$ be given by formula (6.1). Define the functions ω_j for $j = 1, \dots, N_0$ by formula (6.6) and put $\omega_{N_0+1} = \kappa_+ v_+$, $\omega_{N_0+2} = \kappa_- v_-$. Then the function

$$\tilde{\omega}(\mu) = \omega(\mu) - \sum_{n=1}^{N_0+2} \omega_n(\mu) \quad (6.8)$$

has no jumps so that $\tilde{\omega} \in C(\mathbb{T})$. Moreover, it follows from condition (1.3) that

$$\tilde{\omega}(\mu) - \tilde{\omega}(a) = O(|\ln |\mu - a||^{-\beta_0}), \quad \mu \rightarrow a,$$

for all singular points $a = \pm 1, a_j, \bar{a}_j$. Therefore the operator $Q^{-1}H(\tilde{\omega})Q^{-1}$ is compact according to Lemma 6.9.

It follows from formula (5.17) that

$$H(\omega_j) = \kappa_j H_{\varphi_j, \theta_j}^{(0)} + \kappa_j \tilde{H}_{\varphi_j, \theta_j}, \quad j = 1, \dots, N_0, \quad (6.9)$$

where the operators $H_{\varphi_j, \theta_j}^{(0)}$ and $\tilde{H}_{\varphi_j, \theta_j}$ are defined by relations (5.18) and (5.19), respectively. Since $\tilde{\Sigma}_{\varphi_j, \theta_j} \in C^\delta(\mathbb{T})$ (for any $\delta < 1$), Lemma 6.9 implies that the operators $Q^{-1}\tilde{H}_{\varphi_j, \theta_j}Q^{-1}$, are compact.

Comparing definitions (2.4) and (6.8), we see that

$$\tilde{H} = H(\omega) - \sum_{n=1}^{N_0+2} H_n = H(\tilde{\omega}) + \sum_{n=1}^{N_0+2} (H(\omega_n) - H_n).$$

Recall that $H(\omega_n) = H_n$ for $n = N_0 + 1, N_0 + 2$ and $H(\omega_j) - H_j = \kappa_j \tilde{H}_{\varphi_j, \theta_j}$ for $j = 1, \dots, N_0$ according to (6.9). Thus the operator $Q^{-1}\tilde{H}Q^{-1}$ is also compact which verifies Assumption 2.3b.

To check Assumption 2.3c, we have to show that the operators

$$Q^{-1}H(v_+)H(v_-)Q^{-1}, \quad Q^{-1}H(v_\pm)H_jQ^{-1} \quad \text{and} \quad Q^{-1}H_jH_lQ^{-1}, \quad (6.10)$$

where $j, l = 1, \dots, N_0$, $j \neq l$, are compact. For the first of these operators, this statement follows from Lemma 6.10 because the singularities of the symbols v_+ and v_- (located at the points 1 and -1) are separated. According to (6.9), we have

$$H(v_\pm)H_j = H(v_\pm)H(\omega_j) - \kappa_j H(v_\pm)\tilde{H}_{\varphi_j, \theta_j}.$$

The operators $Q^{-1}H(v_\pm)H(\omega_j)Q^{-1}$ are compact again by Lemma 6.10 because the singularities of the symbols v_\pm and ω_j (located at the points ± 1 and a_j, \bar{a}_j) are separated. Observe that

$$Q^{-1}H(v_\pm)\tilde{H}_{\varphi_j, \theta_j}Q^{-1} = (Q^{-1}H(v_\pm)Q)(Q^{-1}\tilde{H}_{\varphi_j, \theta_j}Q^{-1}). \quad (6.11)$$

In the right-hand side, the first factor is bounded by Lemma 6.8 and the second factor is compact by Lemma 6.9. Finally, using again (6.9) we find that

$$H_jH_l = (H(\omega_j) - \kappa_j \tilde{H}_{\varphi_j, \theta_j})(H(\omega_l) - \kappa_l \tilde{H}_{\varphi_l, \theta_l}).$$

The operators $Q^{-1}H(\omega_j)H(\omega_l)Q^{-1}$ are compact because the singularities of the symbols ω_j and ω_l (located at the points a_j, \bar{a}_j and a_l, \bar{a}_l) are separated. The terms containing $\tilde{H}_{\varphi_j, \theta_j}$ or $\tilde{H}_{\varphi_l, \theta_l}$ can be considered quite similarly to (6.11). It follows that the third operator (6.10) is also compact.

Finally, Assumption 2.3d is satisfied according to Lemma 6.8.

Thus we have verified Assumption 2.3 with the operators H_n , $n = 1, \dots, N = N_0 + 2$, defined in Theorem 6.4. Therefore Theorems 6.2, 6.3 and 6.4 are direct consequences of Theorems 2.4, 2.5 and 2.6, respectively.

6.4. Let us now reformulate the results of subs. 6.1 in terms of Hankel operators (3.8) acting in the space $\mathcal{H} = \mathbb{H}_+^2(\mathbb{R})$. We recall that the operators $\mathbf{H} = \mathbf{H}(\psi)$ and $H = H(\omega)$ are unitarily equivalent (see formula (3.8)) if their symbols $\psi(\nu)$ and $\omega(\mu)$ are related by equality (3.7). If the symbol $\omega(\mu)$ has a jump $\varkappa(1)$ or $\varkappa(-1)$ at the point $+1$ or -1 , then the symbol $\psi(\nu)$ has the jump

$$\varkappa(\infty) := \psi(+\infty) - \psi(-\infty) = -\varkappa(1) \quad \text{or} \quad \varkappa(0) := \psi(+0) - \psi(-0) = -\varkappa(-1)$$

at $\nu = \infty$ or $\nu = 0$, respectively. If $\omega(\mu)$ has jumps \varkappa and $-\bar{\varkappa}$ at points a and \bar{a} , then $\psi(\nu)$ has the jumps $-a\varkappa$ and $\bar{a}\bar{\varkappa}$ at the points

$$b = \frac{i}{2} \frac{1+a}{1-a} \quad \text{and} \quad -b = \frac{i}{2} \frac{1+\bar{a}}{1-\bar{a}}.$$

Note that $b < 0$ if $\text{Im } a > 0$.

We assume that a symbol $\psi : \mathbb{R} \rightarrow \mathbb{C}$ satisfies the self-adjointness condition $\psi(-\nu) = \overline{\psi(\nu)}$ and is continuous apart from some jump discontinuities at finitely many points 0 and $b_1, -b_1, \dots, b_{N_0}, -b_{N_0}$ (we suppose that $\overline{b_j} < 0$) with jumps $\varkappa(0)$ and $\varkappa(b_1), \varkappa(-b_1) = -\overline{\varkappa(b_1)}, \dots, \varkappa(b_{N_0}), \varkappa(-b_{N_0}) = -\overline{\varkappa(b_{N_0})}$, respectively. We also suppose that the limits $\psi(\pm\infty)$ exist and are finite. At every singular point b , we assume the logarithmic Hölder continuity of $\psi(\nu)$. It is defined exactly as in (1.3) for finite b and by the relation

$$\psi(\nu) - \psi(\pm\infty) = O(|\ln |\nu||^{-\beta_0}), \quad \nu \rightarrow \pm\infty, \quad \beta_0 > 0,$$

at infinity.

Similarly to (6.1), we set

$$\varkappa(\infty) = 2i\kappa_\infty, \quad \varkappa(0) = 2i\kappa_0, \quad \varkappa(b_j) = 2\kappa_j e^{i\phi_j}, \quad \kappa_j > 0, \quad j = 1, \dots, N_0, \quad (6.12)$$

and

$$\Delta_0 = [0, \kappa_0], \quad \Delta_\infty = [0, \kappa_\infty], \quad \Delta_j = [-\kappa_j, \kappa_j]. \quad (6.13)$$

Then Theorem 6.2 remains true (with the same conditions on β_0) for the operator \mathbf{H} if the numbers κ_+ and κ_- are replaced by $-\kappa_\infty$ and $-\kappa_0$, respectively. In particular, the operator $\mathbf{H}^{(\text{ac})}$ is unitarily equivalent to the orthogonal sum

$$\mathbf{A}_{\Delta_0} \oplus \mathbf{A}_{\Delta_\infty} \oplus \bigoplus_{j=1}^{N_0} \mathbf{A}_{\Delta_j}. \quad (6.14)$$

For the proof, it suffices to notice that under our assumptions on the symbol $\psi(\nu)$, the symbol $\omega(\mu)$ defined by equality (3.7) satisfies Assumption 6.1. Therefore Theorem 6.2 applies to the operator $H(\omega)$ and we only have to use that $\mathbf{H}(\psi) = \mathcal{U}H(\omega)\mathcal{U}^*$ where the unitary operator \mathcal{U} is defined by (3.5).

Theorem 6.3 also remains unchanged if the function $q(\mu)$ is replaced by a bounded function $\mathbf{q}(\nu)$ such that $\mathbf{q}(\nu) = O(|\ln |\nu - b||^{-\beta})$ as $\nu \rightarrow b$ for some $\beta > 1$ in all finite points b of discontinuity of $\psi(\nu)$ and $\mathbf{q}(\nu) = O(|\ln |\nu||^{-\beta})$ as $|\nu| \rightarrow \infty$ if $\psi(+\infty) \neq \psi(-\infty)$.

The reformulation of Theorem 6.7 for Hankel operators $\mathbf{H} = \mathbf{H}(\psi)$ in the space $\mathbb{H}_+^2(\mathbb{R})$ is quite obvious. Now we introduce symbols ψ_0, ψ_∞ and $\psi_j, j = 1, \dots, N_0$, with the same jumps as the symbol ψ at its singular points $0, \infty$ and $(b_j, -b_j), j = 1, \dots, N_0$, respectively. Then again all the assertions of Theorem 2.6 with $N = N_0 + 2$ are true for the operators $H_j = H(\psi_j), j = 1, \dots, N_0$, and $H_{N_0+1} = H(\psi_\infty), H_{N_0+2} = H(\psi_0)$.

The symbols ψ_0, ψ_∞ and ψ_j can be expressed in terms of the function $\zeta(\nu)$ defined by equality (4.6):

$$\psi_0(\nu) = \varkappa(0)\zeta(\nu), \quad \psi_\infty(\nu) = -\varkappa(\infty)\zeta(-\nu^{-1})$$

and

$$\psi_j(\nu) = \varkappa(b_j)\zeta(\nu - b_j) + \varkappa(-b_j)\zeta(\nu + b_j), \quad j = 1, \dots, N_0.$$

Finally, we note that all model operators in the space $\mathbb{H}_+^2(\mathbb{T})$ can be transplanted into the space $\mathbb{H}_+^2(\mathbb{R})$ by the unitary transformation \mathcal{U} . This leads to the reformulation of Theorem 6.4.

6.5. All our results can be extended to Hankel operators acting in spaces of vector-valued functions. Consider, for example, a Hankel operator $H(\omega)$ in the space $\mathbb{H}_+^2(\mathbb{T}) \otimes \mathcal{N}$ where \mathcal{N} is an auxiliary Hilbert space and the symbol $\omega(\mu) : \mathcal{N} \rightarrow \mathcal{N}$ is an operator-valued function. We suppose that $\omega(\mu)$ are compact operators in \mathcal{N} satisfying the condition $\omega(\bar{\mu}) = \omega(\mu)^*$. Then the operator $H(\omega)$ is self-adjoint. The function $\omega(\mu)$ is supposed to be continuous in the operator norm apart from some jump singularities. At singular points $1, -1, a_1, \bar{a}_1, \dots, a_{N_0}, \bar{a}_{N_0}$, we assume condition (1.3). If $\mathcal{N} = \mathbb{C}^k$, then $\omega(\mu)$ is of course a matrix-valued function.

At the points ± 1 , the function $\omega(\mu)$ may have jumps $\varkappa(\pm 1) = 2iK_\pm$ where K_\pm are self-adjoint operators in the space \mathcal{N} . In general, the operators K_\pm have both positive and negative eigenvalues denoted $\kappa_\pm^{(1)}, \kappa_\pm^{(2)}, \dots$.

If $a_j, \text{Im } a_j > 0$, is a complex singular point of $\omega(\mu)$ with a jump $\varkappa(a_j) = 2K_j$, then $\omega(\mu)$ also has the jump $-2K_j^*$ at the conjugate point \bar{a}_j . Let us put $K_j = R_j + iS_j$ where $R_j = R_j^*, S_j = S_j^*$ and construct auxiliary compact self-adjoint operators

$$\mathbf{K}_j = \begin{pmatrix} S_j & -R_j \\ -R_j & -S_j \end{pmatrix} \quad (6.15)$$

in the space $\mathcal{N} \otimes \mathbb{C}^2$. Since $\mathbf{K}_j \mathcal{J} = -\mathcal{J} \mathbf{K}_j$ for the involution $\mathcal{J} = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}$, eigenvalues of the operators \mathbf{K}_j are symmetric with respect to the point 0. We write them as $\pm \kappa_j^{(1)}, \pm \kappa_j^{(2)}, \dots$. Diagonalizing matrix (6.15), we find that

$$\mathbf{K}_j = Y_j^* \text{diag}\{\kappa_j^{(1)}, -\kappa_j^{(1)}, \kappa_j^{(2)}, -\kappa_j^{(2)}, \dots\} Y_j$$

where Y_j are unitary operators in the space \mathcal{N} . Of course, the explicit expression (5.10) for these operators no longer makes sense.

Instead of (6.2), we now have the set of intervals

$$\Delta_{\pm}^{(l)} = [0, \kappa_{\pm}^{(l)}] \quad \text{and} \quad \Delta_j^{(l)} = [-\kappa_j^{(l)}, \kappa_j^{(l)}], \quad l = 1, 2, \dots$$

Then Theorem 6.2 remains true with the following natural modifications. In definition (6.3) one should set

$$A_{\Delta_{\pm}} = \bigoplus_l A_{\Delta_{\pm}^{(l)}} \quad \text{and} \quad A_{\Delta_j} = \bigoplus_l A_{\Delta_j^{(l)}}$$

which entails also an obvious modification of equality (1.4). Eigenvalues of $H(\omega)$, distinct from 0, $\kappa_{\pm}^{(l)}$ and $\pm\kappa_j^{(l)}$, $j = 1, \dots, N_0$, for all values of l , have finite multiplicities and can accumulate only to these points. Theorem 6.3 remains unchanged.

Let us discuss a generalization of Theorem 6.4. Similarly to the scalar case (cf. Section 4), the model operators corresponding to the jumps at the points $\mu = 1$ and $\mu = -1$ can be defined by the formulas

$$H_{N_0+1} = H(v_+) \otimes K_+ \quad \text{and} \quad H_{N_0+2} = H(v_-) \otimes K_-.$$

Then the proof of Theorem 4.5 works directly because the operators $H(v_{\pm})$ and K_{\pm} act in different spaces ($\mathbb{H}_+^2(\mathbb{T})$ and \mathcal{N} , respectively). Diagonalizing the self-adjoint operator K_{\pm} , we reduce the problem to the scalar case. We emphasize that the space \mathcal{N} entering into Definition 2.1 is the same as here.

The model operators H_j corresponding to the jumps at the points (a_j, \bar{a}_j) are constructed similarly to Section 5. The role of the model symbols $\kappa\omega_{\varphi}(\mu)$ and $\kappa\Sigma_{\varphi}^{(0)}(\mu)$ (recall formulas (5.5) and (5.7)) for jumps at the points $(i - i)$ is now played by the matrix-values functions

$$\omega_K(\mu) = (S - \mu R)v(\mu^2).$$

and $Kv(\mu)$, respectively. Therefore the model operators H_j corresponding to the jumps at arbitrary complex points (a_j, \bar{a}_j) are constructed by the formula

$$H_j = T_{\alpha_j}^* U^* H(K_j v) U T_{\alpha_j}.$$

This generalizes the “scalar” formula $H_j = \kappa_j H_{\varphi_j, \theta_j}^{(0)}$ where $H_{\varphi_j, \theta_j}^{(0)}$ are operators (5.18).

7. INFINITE MATRICES AND INTEGRAL OPERATORS

The results of Section 6 can be reformulated in terms of Hankel operators acting in the spaces ℓ_+^2 and $L^2(\mathbb{R}_+)$. This requires the Fourier expansion of symbols $\omega(\mu)$, $\mu \in \mathbb{T}$, and the Fourier transform of symbols $\psi(\nu)$, $\nu \in \mathbb{R}$. Therefore the results stated in terms of matrix elements h_n of operators \widehat{H} and of kernels $\mathbf{h}(t)$ of operators $\widehat{\mathbf{H}}$ are necessarily rather far from optimal. In notation of Section 3, we have $h_n = \hat{\omega}_n$ and $\mathbf{h}(t) = (2\pi)^{-1/2} \hat{\psi}(t)$.

7.1. Let us consider the space $\mathcal{H} = \ell_+^2$ where a Hankel operator \widehat{H} acts by the formula (1.5) with matrix elements h_n . We study the case of matrix elements with asymptotics (1.6). To use the results of the previous section, we have to construct a symbol ω satisfying Assumption 6.1 and such that $\widehat{H} = \mathcal{F}H(\omega)\mathcal{F}^*$. Let us start with an elementary observation.

Lemma 7.1. *Suppose that*

$$\tilde{h}_n = O(n^{-1}(\ln n)^{-\alpha}), \quad \alpha > 1.$$

Then the function

$$\tilde{\omega}(\mu) = \sum_{n=0}^{\infty} \tilde{h}_n \mu^n$$

satisfies the logarithmic Hölder condition with exponent $\beta = \alpha - 1$, that is,

$$|\tilde{\omega}(\mu') - \tilde{\omega}(\mu)| \leq C(1 + |\ln |\mu' - \mu||)^{-\beta}, \quad \mu, \mu' \in \mathbb{T}. \quad (7.1)$$

Proof. For all N , we have

$$|\tilde{\omega}(\mu') - \tilde{\omega}(\mu)| \leq \sum_{n \leq N} |(\mu')^n - \mu^n| |\tilde{h}_n| + \sum_{n > N} |(\mu')^n - \mu^n| |\tilde{h}_n|. \quad (7.2)$$

The first sum here is bounded by

$$\sum_{n \leq N} n |\mu' - \mu| |\tilde{h}_n| \leq |\mu' - \mu| N \sum_{n=0}^{\infty} |\tilde{h}_n|.$$

The second sum in (7.2) is bounded by

$$2 \sum_{n > N} |\tilde{h}_n| \leq C \sum_{n > N} n^{-1} (\ln n)^{-\alpha} \leq C_1 (\ln N)^{-\beta}.$$

Choosing, for example, $N = |\mu' - \mu|^{-1/2}$ and substituting these two estimates into (7.2), we get (7.1). \square

Let us introduce the Hankel matrices \widehat{H}_+ and \widehat{H}_- with elements

$$h_n^{(+)} = \pi^{-1}(n+1)^{-1} \quad \text{and} \quad h_n^{(-)} = (-1)^n \pi^{-1}(n+1)^{-1}, \quad (7.3)$$

respectively. As can easily be checked by a direct calculation, the Fourier coefficients of the function

$$\omega_+(\mu) = i(1 - \psi/\pi)e^{-i\psi}, \quad \mu = e^{i\psi}, \quad 0 \leq \psi \leq 2\pi, \quad (7.4)$$

equal $h_n^{(+)}$ if $n \geq 0$. Similarly, the Fourier coefficients of the function $\omega_-(\mu) = \omega_+(-\mu)$ equal $h_n^{(-)}$ if $n \geq 0$. It follows that $\widehat{H}_{\pm} = \mathcal{F}H(\omega_{\pm})\mathcal{F}^*$. Note that $\omega_{\pm}(\bar{\mu}) = \overline{\omega_{\pm}(\mu)}$.

Lemma 7.2. ¹⁰ *The operators \widehat{H}_\pm have the a.c. simple spectra coinciding with the interval $[0, 1]$. They have no singular continuous spectra and their eigenvalues distinct from 0 and 1 have finite multiplicities and may accumulate to these points only.*

²⁰ *Let the symbols v_\pm be defined by equalities (4.6), (4.8) and (4.11). Put $\widehat{H}(v_\pm) = \mathcal{F}H(v_\pm)\mathcal{F}^*$. Then the wave operators $W_+(\widehat{H}_\pm, \widehat{H}(v_\pm))$ and $W_-(\widehat{H}_\pm, \widehat{H}(v_\pm))$ exist and are complete.*

Proof. Consider, for example, the operators \widehat{H}_+ and $\widehat{H}(v_+)$. Note that the functions $\omega_+(\mu)$ and $v_+(\mu)$ are smooth on $\mathbb{T} \setminus \{1\}$ and $\omega_+(1 \pm i0) = v_+(1 \pm i0) = \pm i$. Therefore $\omega_+ - v_+ \in C^\delta(\mathbb{T})$ (for any $\delta < 1$) so that Theorems 6.2 and 6.4 (see, in particular, Corollary 6.5) apply to the operators $H(v_+)$ and $H(\omega_+)$. Then it remains to transplant the results obtained into the space ℓ_+^2 by the operator \mathcal{F} . \square

Remark 7.3. Actually, the operators \widehat{H}_\pm have no eigenvalues, and they can be explicitly diagonalized. Indeed, as shown in papers [7, 16], the operator $\widehat{\mathbf{H}}_+ = \mathcal{L}\widehat{H}_+\mathcal{L}^*$ is the Hankel integral operator in $L^2(\mathbb{R}_+)$ with kernel $h_+(t) = \pi^{-1}t^{-1}e^{-t}$; it can be diagonalized in terms of the Whittaker functions. The operator \widehat{H}_- has the same properties since $\widehat{H}_- = \widehat{R}^*\widehat{H}_+\widehat{R}$ where \widehat{R} is the unitary operator in ℓ_+^2 defined by $(\widehat{R}u)_n = (-1)^n u_n$.

Next, we consider the Hankel matrix $\widehat{H}_{\theta,\varphi}$ with elements

$$h_n(\theta, \varphi) = 2\pi^{-1}(n+1)^{-1} \sin(n\theta - \varphi).$$

In contrast to the operators \widehat{H}_\pm , we cannot diagonalize the operators $\widehat{H}_{\theta,\varphi}$ explicitly. Nevertheless similarly to Lemma 7.2, we can describe the structure of their spectra.

Lemma 7.4. *For all θ and φ , the operators $\widehat{H}_{\theta,\varphi}$ have the a.c. simple spectra coinciding with the interval $[-1, 1]$. They have no singular continuous spectra and their eigenvalues distinct from 0, 1 and -1 have finite multiplicities and may accumulate to these points only.*

Proof. Set

$$\omega_{\theta,\varphi}(\mu) = i(\omega_+(e^{-i\theta}\mu)e^{i\varphi} - \omega_+(e^{i\theta}\mu)e^{-i\varphi})$$

where the function ω_+ is defined by equality (7.4). The function $\omega_{\theta,\varphi}(\mu)$ satisfies Assumption 6.1. It has only two points $e^{i\theta}$ and $e^{-i\theta}$ of discontinuity with the jumps $-2e^{i\varphi}$ and $2e^{-i\varphi}$, respectively. Therefore Theorem 6.2 (see, in particular, Corollary 6.6) applies to the operator $H(\omega_{\theta,\varphi})$.

Using expression (7.3) for the Fourier coefficients of the function $\omega_+(\mu)$, we see that the Fourier coefficients of the function $\omega_{\theta,\varphi}(\mu)$ equal

$$\hat{\omega}_n(\theta, \varphi) = i(e^{i\varphi}e^{-in\theta} - e^{-i\varphi}e^{in\theta})h_n^{(+)} = h_n(\theta, \varphi), \quad n \geq 0,$$

and hence $\widehat{H}_{\theta,\varphi} = \mathcal{F}H(\omega_{\theta,\varphi})\mathcal{F}^*$. \square

Let us return to the operator \widehat{H} whose matrix elements h_n have asymptotics (1.6) where $\alpha_0 > 2$. Put

$$\tilde{h}_n = h_n - \kappa_+ h_n^{(+)} - \kappa_- h_n^{(-)} - \sum_{j=1}^{N_0} \kappa_j h_n(\theta_j, \varphi_j), \quad n \geq 0,$$

Since $\tilde{h}_n = O(n^{-1}(\ln n)^{-\alpha_0})$ as $n \rightarrow \infty$, it follows from Lemma 7.1 that the function $\tilde{\omega} = \mathcal{F}^* \tilde{h}$ is logarithmic Hölder continuous with exponent $\beta_0 = \alpha_0 - 1 > 1$. Set

$$\omega(\mu) = \kappa_+ \omega_+(\mu) + \kappa_- \omega_-(\mu) + \sum_{j=1}^{N_0} \kappa_j \omega_{\theta_j, \varphi_j}(\mu) + \tilde{\omega}(\mu).$$

By our construction, the Fourier coefficients of this function are $(\mathcal{F}\omega)_n = h_n$, $n \geq 0$. Let us now apply the results of subs. 6.1 to the operators $H(\omega)$ and $H_j = \kappa_j H(\omega_{\theta_j, \varphi_j})$, $j = 1, \dots, N_0$, $H_{N_0+1} = \kappa_+ H(\omega_+)$, $H_{N_0+2} = \omega_- H(\omega_-)$. Transplanting these results into the space ℓ_+^2 by the operator \mathcal{F} , we obtain the following assertion.

Theorem 7.5. *Suppose that coefficients h_n of a Hankel matrix \widehat{H} admit representation (1.6) where θ_j are distinct numbers in $(0, \pi)$; the phases $\varphi_j \in [0, \pi)$ and the amplitudes $\kappa_+, \kappa_-, \kappa_j \in \mathbb{R}$ are arbitrary.*

1⁰ If $\alpha_0 > 2$, then the operator $\widehat{H}^{(\text{ac})}$ is unitarily equivalent to the orthogonal sum (6.3). Moreover, the wave operators $W_{\pm}(\widehat{H}, \kappa_{\pm} \widehat{H}_{\pm})$ and $W_{\pm}(\widehat{H}, \kappa_j \widehat{H}(\theta_j, \varphi_j))$, $j = 1, \dots, N_0$, exist, their ranges are mutually orthogonal, and their orthogonal sum exhausts the subspace $\mathcal{H}^{(\text{ac})}(\widehat{H})$.

2⁰ If $\alpha_0 > 3$, then the singular continuous spectrum of \widehat{H} is empty and its eigenvalues different from the points 0, κ_+ , κ_- and $\pm \kappa_j$ have finite multiplicities and may accumulate only to these points.

7.2. Next, we consider a Hankel operator $\widehat{\mathbf{H}}$ acting in the space $\mathcal{H} = L^2(\mathbb{R}_+)$ by the formula

$$(\widehat{\mathbf{H}}u)(t) = \int_0^\infty \mathbf{h}(t+s)u(s)ds.$$

We suppose that $\mathbf{h} \in L_{\text{loc}}^1(\mathbb{R}_+)$ and $\mathbf{h}(t) = O(t^{-1})$ as $t \rightarrow \infty$ and $t \rightarrow 0$. The operator $\widehat{\mathbf{H}}$ is symmetric if $\mathbf{h}(t)$ is a real function. Observe that $\widehat{\mathbf{H}}$ is compact if $\mathbf{h}(t) = o(t^{-1})$ as $t \rightarrow \infty$ and $t \rightarrow 0$. On the other hand, if $\mathbf{h}(t)$ behaves as t^{-1} for $t \rightarrow \infty$ or for $t \rightarrow 0$, then the operator $\widehat{\mathbf{H}}$ acquires an a.c. spectrum. For example, the Mehler operator \mathcal{M} defined by formula (4.1) has the simple a.c. spectrum coinciding with $[0, 1]$.

We consider kernels with singularities both at $t = \infty$ and $t = 0$. To be precise, we assume that

$$\mathbf{h}(t) = (\pi t)^{-1} \left(h_\infty + 2 \sum_{j=1}^{N_0} h_j \sin(b_j t - \phi_j) + O(|\ln t|^{-\alpha_0}) \right), \quad t \rightarrow \infty, \quad (7.5)$$

and

$$\mathbf{h}(t) = (\pi t)^{-1}(h_0 + O(|\ln t|^{-\alpha_0})), \quad t \rightarrow 0. \quad (7.6)$$

We emphasize that the right-hand side of (7.5) contains oscillating terms.

As in the previous subsection, we are going to use the results of Section 6 on Hankel operators in the Hardy space $\mathbb{H}_+^2(\mathbb{R})$ (see subs. 6.4). Thus we have to construct a piecewise continuous symbol $\psi(\nu)$ such that $\widehat{\mathbf{H}} = \Phi \mathbf{H}(\psi) \Phi^*$. The following assertion plays the role of Lemma 7.1.

Lemma 7.6. *Suppose that a function $\widetilde{\mathbf{h}} \in L^1(\mathbb{R}_+)$ obeys the condition*

$$\widetilde{\mathbf{h}}(t) = O(t^{-1}|\ln t|^{-\alpha}), \quad \alpha > 1, \quad (7.7)$$

as $t \rightarrow \infty$. Then the function

$$\tilde{\psi}(\nu) = \int_0^\infty \widetilde{\mathbf{h}}(t) e^{i\nu t} dt$$

is logarithmic Hölder continuous with exponent $\beta = \alpha - 1$, that is,

$$|\tilde{\psi}(\nu') - \tilde{\psi}(\nu)| \leq C(1 + |\ln |\nu' - \nu||)^{-\beta}, \quad \nu, \nu' \in \mathbb{R}. \quad (7.8)$$

Proof. It follows from condition (7.7) that for an arbitrary $a > 1$

$$\left| \int_0^a \widetilde{\mathbf{h}}(t)(e^{i\nu't} - e^{i\nu t}) dt \right| \leq C|\nu' - \nu| \int_0^a t |\widetilde{\mathbf{h}}(t)| dt \leq C_1 |\nu' - \nu| a \int_0^\infty |\widetilde{\mathbf{h}}(t)| dt. \quad (7.9)$$

Moreover, we have

$$\left| \int_a^\infty \widetilde{\mathbf{h}}(t)(e^{i\nu't} - e^{i\nu t}) dt \right| \leq 2 \int_a^\infty |\widetilde{\mathbf{h}}(t)| dt \leq C |\ln a|^{-\beta}.$$

Combining this estimate with (7.9) and choosing, for example, $a = |\nu' - \nu|^{-1/2}$, we get (7.8). \square

Let us first consider kernels $\mathbf{h}(t)$ with asymptotics (7.5) as $t \rightarrow \infty$ and regular at the point $t = 0$. Recall that, as shown in subs. 4.2, the symbol $\psi_\infty(\nu)$ of the Mehler operator $\mathcal{M} =: \widehat{\mathbf{H}}_\infty$ can be chosen as $\psi_\infty(\nu) = 2i\zeta(\nu)$ where $\zeta(\nu)$ is function (4.6). It follows that the Fourier transform of the function

$$\psi_{\phi,b}(\nu) = 2e^{-i\phi}\zeta(\nu+b) - 2e^{i\phi}\zeta(\nu-b)$$

equals $(2\pi)^{1/2}\mathbf{h}_{\phi,b}(t)$ where

$$\mathbf{h}_{\phi,b}(t) = 2\pi^{-1}(2+t)^{-1}\sin(bt-\phi).$$

Thus, a symbol of the operator $\widehat{\mathbf{H}}_{\phi,b}$ with integral kernel $\mathbf{h}_{\phi,b}(t)$ can be chosen as $\psi_{\phi,b}(\nu)$. Since the function $\psi_{\phi,b}(\nu)$ has only two jumps $-2e^{i\phi}$ and $2e^{-i\phi}$ at the points b and $-b$, respectively, Theorem 6.2 entails the following result (cf. Lemma 7.4).

Lemma 7.7. *For all ϕ and $b \neq 0$, the operators $\widehat{\mathbf{H}}_{\phi,b}$ have the a.c. simple spectra coinciding with the interval $[-1, 1]$. They have no singular continuous spectra and their eigenvalues distinct from 0, 1 and -1 have finite multiplicities and may accumulate to these points only.*

It remains to construct a model operator for the kernel with singularity (7.6) as $t \rightarrow 0$. Observe that the Fourier transform of the function

$$\psi_0(\nu) = 2\pi^{-1}i \int_0^\infty \frac{\sin(t\nu)}{t} e^{-t} dt = \pi^{-1} \text{v.p.} \int_{-\infty}^\infty \frac{e^{it\nu}}{t} e^{-|t|} dt$$

(the right integral is understood in the sense of the principal value) equals

$$\hat{\psi}_0(t) = (2/\pi)^{1/2} \text{v.p.} t^{-1} e^{-|t|}.$$

This implies that $\psi_0(\nu)$ is a symbol of the Hankel operator $\widehat{\mathbf{H}}_0$ with kernel $\mathbf{h}_0(t) = (\pi t)^{-1} e^{-t}$. The function $\psi_0(\nu)$ is smooth, but its limits at infinity $\psi_0(\pm\infty) = \pm i$ are different. Thus by Corollary 6.5, the operator $\widehat{\mathbf{H}}_0$ has the a.c. simple spectrum coinciding with the interval $[0, 1]$. It has no singular continuous spectrum and its eigenvalues distinct from 0 and 1 have finite multiplicities and may accumulate to these points only. Actually, it is known (see Remark 7.3) that the operator $\widehat{\mathbf{H}}_0$ has no eigenvalues, and it can be explicitly diagonalized.

Let us return to the operator $\widehat{\mathbf{H}}$ whose kernel has asymptotics (7.5) and (7.6). Put $\mathbf{h}_\infty(t) = h_\infty \pi^{-1} (t+2)^{-1}$, $\mathbf{h}_j(t) = 2h_j \pi^{-1} (t+2)^{-1} \sin(b_j t - \phi_j)$, $\mathbf{h}_0(t) = h_0 (\pi t)^{-1} e^{-t}$ and

$$\tilde{\mathbf{h}}(t) = \mathbf{h}(t) - \mathbf{h}_\infty(t) - \sum_{j=1}^{N_0} \mathbf{h}_j(t) - \mathbf{h}_0(t). \quad (7.10)$$

By our construction, $\tilde{\mathbf{h}} \in L^1(\mathbb{R}_+)$ and it satisfies condition (7.7) with $\alpha = \alpha_0$ both as $t \rightarrow \infty$ and $t \rightarrow 0$. It follows from Lemma 7.6 that the function

$$\tilde{\psi}(\nu) = \int_0^\infty \tilde{\mathbf{h}}(t) e^{i\nu t} dt \quad (7.11)$$

is logarithmic Hölder continuous with exponent $\beta_0 = \alpha_0 - 1$. Of course $\tilde{\psi}(\nu) \rightarrow 0$ as $|\nu| \rightarrow \infty$, but to verify the condition

$$\tilde{\psi}(\nu) = O(|\ln |\nu||^{-\beta_0}) \quad \text{as } |\nu| \rightarrow \infty, \quad (7.12)$$

we need an additional (very weak) assumption.

Assumption 7.8. *A function $\mathbf{h}(t)$ is absolutely continuous except a finite number of jumps and, for some k ,*

$$\int_{a^{-1}}^a |\mathbf{h}'(t)| dt = O(a^k) \quad \text{as } a \rightarrow \infty.$$

Lemma 7.9. *Let conditions (7.5), (7.6) and Assumption 7.8 be satisfied. Then function (7.11) obeys estimate (7.12) with $\beta_0 = \alpha_0 - 1$.*

Proof. By definition (7.10), the function $\tilde{\mathbf{h}}(t)$ also satisfies Assumption 7.8. Integrating by parts, we see that

$$\left| \int_{a^{-1}}^a \tilde{\mathbf{h}}(t) e^{i\nu t} dt \right| \leq C |\nu|^{-1} a^k.$$

where $a \rightarrow \infty$. It follows from estimate (7.7) on $\tilde{\mathbf{h}}(t)$ for $t \rightarrow 0$ and $t \rightarrow \infty$ that

$$\left| \int_0^{a^{-1}} \tilde{\mathbf{h}}(t) e^{i\nu t} dt \right| + \left| \int_a^\infty \tilde{\mathbf{h}}(t) e^{i\nu t} dt \right| \leq C |\ln a|^{-\beta_0}.$$

Choosing, for example, $a = |\nu|^{1/(2k)}$, we obtain (7.12). \square

Let us now put

$$\psi(\nu) = h_\infty \psi_\infty(\nu) + \sum_{j=1}^{N_0} h_j \psi_{\phi_j, b_j}(\nu) + h_0 \psi_0(\nu) + \tilde{\psi}(\nu).$$

It follows from formula (7.10) that $\hat{\mathbf{H}} = \Phi \mathbf{H}(\psi) \Phi^*$. Now we can apply the results of subs. 6.4 to the operators $\mathbf{H}(\psi)$ and $h_\infty \mathbf{H}(\psi_\infty)$, $h_j \mathbf{H}(\psi_{\phi_j, b_j})$, $h_0 \mathbf{H}(\psi_0)$. This yields the following result.

Theorem 7.10. *Let $\hat{\mathbf{H}}$ be the Hankel operator with kernel $\mathbf{h} \in L^1_{\text{loc}}(\mathbb{R}_+)$ satisfying conditions (7.5), (7.6) and Assumption 7.8. We suppose that the numbers $b_1, \dots, b_{N_0} \in \mathbb{R}_+ \setminus \{0\}$ are distinct, the phases $\phi_j \in [0, \pi)$, $j = 1, \dots, N_0$, as well as the amplitudes $h_n \in \mathbb{R}$, $n = 1, \dots, N$, are arbitrary.*

1⁰ If $\alpha_0 > 2$, then the operator $\hat{\mathbf{H}}^{(\text{ac})}$ is unitarily equivalent to the orthogonal sum

$$\mathbf{A}_{(0, h_0)} \oplus \mathbf{A}_{(0, h_\infty)} \oplus \bigoplus_{j=1}^{N_0} \mathbf{A}_{(-h_j, h_j)}. \quad (7.13)$$

Moreover, the wave operators $W_\pm(\hat{\mathbf{H}}, h_0 \hat{\mathbf{H}}_0)$, $W_\pm(\hat{\mathbf{H}}, h_\infty \hat{\mathbf{H}}_\infty)$ and $W_\pm(\hat{\mathbf{H}}, h_j \hat{\mathbf{H}}(\theta_j, \varphi_j))$, $j = 1, \dots, N_0$, exist, their ranges are mutually orthogonal, and their orthogonal sum exhausts the subspace $\mathcal{H}^{(\text{ac})}(\hat{\mathbf{H}})$.

2⁰ If $\alpha_0 > 3$, then the singular continuous spectrum of $\hat{\mathbf{H}}$ is empty and its eigenvalues different from the points 0, h_0 , h_∞ and $\pm h_j$ have finite multiplicities and may accumulate only to these points.

Remark 7.11. If there is no singularity at the point $t = 0$, that is, $\mathbf{h} \in L^1(0, r)$ for $r < \infty$, then condition (7.6) and Assumption 7.8 disappear and the term $\mathbf{A}_{(0, h_0)}$ in (7.13) should be omitted. Indeed, in this case the point $\nu = \infty$ is not singular for the symbol of the operator $\Phi^* \hat{\mathbf{H}} \Phi$ so that we do not need to verify (7.12) and hence condition (7.11) is not required.

Remark 7.12. The case $h_0 = h_\infty$, $h_j = 0$ for all $j = 1, \dots, N_0$ was considered in [19]. It was shown there that the assertion 1^o of Theorem 7.10 holds true for $\alpha_0 > 1$ and the assertion 2^o – for $\alpha_0 > 2$. Assumption 7.8 was also not required. Therefore it can be expected that the conditions of Theorem 7.10 are not optimal. The same remark applies to Theorem 7.5. We also note that using the Mourre method J. S. Howland [5] obtained the spectral results (but not the results about the wave operators) of Theorem 7.10 assuming that $h_j = 0$ for all $j = 1, \dots, N_0$ but admitting that $h_0 \neq h_\infty$.

As far as earlier results about Hankel operators $\hat{\mathbf{H}}$ with oscillating kernels are concerned, we mention the paper [6] where the case $\mathbf{h}(t) = 2(\pi t)^{-1} \sin(bt)$ was considered. Obviously, for different $b > 0$, these operators are unitarily equivalent to each other. With a help of the results of [16], it was proven in [6] that the spectrum of such $\hat{\mathbf{H}}$ is a.c. simple and coincides with the interval $[-1, 1]$. This result is of course consistent with Theorem 7.10.

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